Studying Voting for Committees Using Wreath Products

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Ratliff and Wheaton College

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- Voter preferences can not necessarily be determined by preferences on individuals.
- How do we capture these complex voter preferences without a complete ranking?

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Focus the Attention

- We'll focus solely on committee elections with structure similar to the Wheaton elections.
- One member from each of n departments, each boasting k candidates

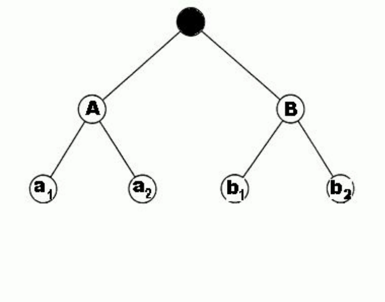
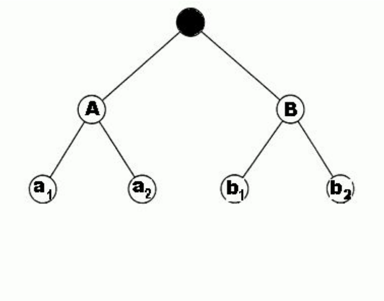


Figure: Simple case with $n = 2$, $k = 2$

Committees!

Keeping Track

$W = a_1 b_1$ $X = a_1 b_2$ $Y = a_2 b_1$ $Z = a_2 b_2$



An Algebraic Approach

- Our profile space is the vector space whose basis is the set of all possible full rankings.

$$\begin{array}{c} WXYZ \\ XZWY \\ ZYXW \\ YWZX \\ XWZY \\ WYXZ \\ YZWX \\ ZXYW \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 3 \\ 3 \\ 1 \\ 3 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

An Algebraic Approach

- A vote-scoring procedure can be realized as a transformation from profile space to a results space.

$$T_{\mathbf{w}} : \mathbb{Q}^{24} \rightarrow \mathbb{Q}^{4}.$$

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$$T_{\textbf{Borda}} : \begin{matrix} WXYZ \\ XZWY \\ ZYXW \\ YWZX \\ XWZY \\ WYXZ \\ YZWX \\ ZXYW \\ \vdots \end{matrix} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 3 \\ 3 \\ 1 \\ 3 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow \begin{matrix} W \\ X \\ Y \\ Z \end{matrix} \begin{bmatrix} 27 \\ 25 \\ 23 \\ 21 \end{bmatrix}$$

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- Symmetric group decomposition of profile spaces is already understood.
- Try a wreath product! In particular, $S_k[S_n]$.

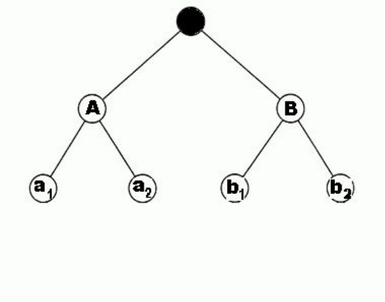

Figure: Try $S_2[S_2]$!

Why the Wreath Product?

Restrictions

$W = a_1 b_1$ $X = a_1 b_2$ $Y = a_2 b_1$ $Z = a_2 b_2$

The wreath product recognizes only the permutations realizable by switching candidates.

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Impossible Moves

We can not turn full ranking WXYZ into ranking WXZY by changing candidates. To turn Y into Z, we must switch candidates b_1 and b_2... But this would also change W and X!

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Therefore...

Thus, the wreath product gives us a finer decomposition of the profile space!

Why the Wreath Product?

- It's a smaller group, too!

Table: Comparing Wreath and Symmetric for $k = 2$

n	$S_2[S_n]$	S_{2^n}
2	8	24
3	48	40,320
4	384	2.09×10^{13}

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Starting with the Simple

- Consider the simple case with $n = 2$, $k = 2$.
- 24-dimensional profile space decomposes into three 8-dimensional pockets, each an orbit of $S_2[S_2]$.

Starting with the Simple

First Orbit

WXYZ	XZWY	ZYXW	YWZX
XWZY	WYXZ	YZWX	ZXYW

Second Orbit

WXZY	XZYW	ZYWX	YWXZ
XWYZ	WYZX	YZXW	ZXWY

Third Orbit

WZXY	XYZW	ZWYX	YXWZ
XYWZ	WZYX	YXZW	ZWXY

Keeping Track

$W = a_1 b_1$ $X = a_1 b_2$ $Y = a_2 b_1$ $Z = a_2 b_2$

We Break Down So That We May Build!

- We can decompose any of these orbits using the representation theory of $S_2[S_2]$.

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- We can decompose any of these orbits using the representation theory of $S_2[S_2]$.
- The decomposition of one of the pockets illuminated a way to construct a profile which produced different results under Borda and plurality scorings.

And Here It Is!

- With the decomposition, we can build a paradoxical profile by using only one of the three pockets!

Borda Count

$$T_{\textbf{Borda}} : \begin{matrix} WXYZ \\ XZWY \\ ZYXW \\ YWZX \\ XWZY \\ WYXZ \\ YZWX \\ ZXYW \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 3 \\ 3 \\ 1 \\ 3 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \rightarrow \begin{matrix} W \\ X \\ Y \\ Z \end{matrix} \begin{bmatrix} 27 \\ 25 \\ 23 \\ 21 \end{bmatrix}$$

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Plurality

$$T_{\textbf{Plurality}} : \begin{matrix} WXYZ \\ XZWY \\ ZYXW \\ YWZX \\ XWZY \\ WYXZ \\ YZWX \\ ZXYW \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix} \begin{bmatrix} 2 \\ 3 \\ 0 \\ 3 \\ 3 \\ 1 \\ 3 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \rightarrow \begin{matrix} W \\ X \\ Y \\ Z \end{matrix} \begin{bmatrix} 9 \\ 18 \\ 18 \\ 3 \end{bmatrix}$$

Once Again

- It's a MUCH smaller group!

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Future Work

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- Representation theory of $S_2[S_n]$ for all n
- Characterize the profile space decomposition
- Use the algebraic perspective to discuss voting procedures for electing committees under this structure

Thanks!