Optimal Experimental Design for Discriminating Models in Bandit Problems

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Question of Interest:

- Discriminating between competing models for decision-making in the bandit problems
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- Intuitively, we want to design the experiment in such a way that the result will support the correct model to the largest extent
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We follow previous work in optimal experimental design for distinguishing psychological models by Myung and Pitt (to appear in *Psychological Review*)
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▶ each of which has a fixed but unknown reward rate, drawn from a fixed but unknown environmental distribution
▶ to maximize their total number of rewards over a short sequence of trials
▶ the design for experiments in bandit problems is the set of reward rates
Optimal design

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- The maximization or minimization is over the design parameter \( d \in D \)
- The outcome of the experiment is defined by \( p_d(y|\theta) \), i.e. the observables come from a distribution conditional on some unknown parameter vector \( \theta \)
- The model is complete by a prior distribution \( p(\theta) \) for the parameter
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Since the design parameter $d$ has to be chosen before observing the experiment, we need to maximize the expectation of $u(d, \theta, y)$ with respect to $(\theta, y)$. 

The global utility is denoted by $U(d)$. We can formally state the design problem as $d^* = \arg \max_{d \in D} U(d)$, where $U(d) = \int u(d, \theta, y) p(\theta, y) d\theta dy = \int u(d, \theta, y) p(\theta) p(y | \theta) d\theta dy$ (1).
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\begin{align}
U(d) &= \int u(d, \theta, y) p_d(\theta, y) d\theta dy \\
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To start with, we try to find one best pair of reward rates for one game with 8 trials.
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\[ U(d) = p(A) \int BF_{A/B} p(y_A | \gamma, d) p(\gamma) dy_A d\gamma \]

\[ + p(B) \int BF_{B/A} p(y_B | \gamma_w, \gamma_l, d) p(\gamma_w) p(\gamma_l) dy_B d\gamma_w d\gamma_l \]

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BF_{A/B} = \frac{\int p(y_A|\gamma) p(\gamma) d\gamma}{\int \int p(y_A|\gamma_w, \gamma_I) p(\gamma_w) p(\gamma_I) d\gamma_w d\gamma_I}
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With a prior distribution of the model parameters, and the formal mathematical set up of the models, we can solve this optimization task by numeric methods.
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We start with trying various priors for the model parameters to find the optimal design under these priors.
Figure: Utility surface corresponding to $\gamma \sim beta(5, 1)$, $\gamma_w \sim beta(5, 1)$, $\gamma_l \sim beta(4, 2)$. 
Figure: Utility surface corresponding to $\gamma \sim beta(5, 1), \gamma_w \sim beta(1, 1), \gamma_l \sim beta(1, 1)$. 
MCMC Sampling Approach

- We can always use prior simulation Monte Carlo to find the entire utility surface and find the mode.
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- But this is usually expensive

Another approach (Muller, 1999) is to recast the problem as a problem of simulation from an augmented probability model

\[ h(d, \theta, y) \propto u(d, \theta, y) p(\theta) p(y|\theta) \]

Under \( h \), the marginal distribution in \( d \) is proportional to \( U(d) \), i.e., \( h(d) \) is proportional to \( U(d) \), as desired

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Annealing Procedure

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\[
h_J(d, \theta_1, y_1, \ldots, \theta_J, y_J) \propto \prod_{j=1}^{J} u(d, \theta_j, y_j)p(\theta_j)p(y_j|\theta_j) \quad (6)
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- The implied marginal in \( d \) is proportional to the \( J \)-th power of the expected utility, \( h_J(d) \propto U^J(d) \).
Figure: Samples of optimal designs. $\gamma \sim \text{beta}(1, 1)$, $\gamma_w \sim \text{beta}(1, 1)$, $\gamma_I \sim \text{beta}(1, 1)$.

The mode is $p_1 = .50, p_2 = .50$. 
Figure: Samples of optimal designs. $\gamma \sim beta(18.87, 7.791)$, $\gamma_w \sim beta(2.872, 0.680)$, $\gamma_l \sim beta(2.318, 1.818)$. 
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- With probability $\epsilon$, choose randomly
- With probability $1-\epsilon$, choose the alternative with the highest estimated reward rate
- Find optimal design to distinguish this model and WSLS model, using empirical priors for the model parameters
Figure: Samples of optimal designs. $\gamma \sim beta(18.87, 7.791)$, $\epsilon \sim beta(5.8, 11.3)$. 
Discussion

Given quantitative models and prior knowledge about the distribution of model parameters, we are able to design an experiment in order to get data which will discriminating between these models to the largest extent.

Our application of the design optimization algorithm had the following findings:

▶ To discriminate between the two versions of WSLS model, the best type of experimental design is to pick two extreme reward rates, one large and one small.

▶ To discriminate between the WSLS model and $\epsilon$-greedy model, the best type of experimental design is to pick up two moderately small reward rates.

Individual difference in the prior for model parameters, as well as the prior for models, are interesting questions to be addressed in future works.
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Thank You!
We assess this integral by prior simulation Monte Carlo. With prior knowledge about the model parameters, for one \( p_i \) pair, i.e. one design \( d \),

1. draw \( \gamma \) from \( p(\gamma) \); draw \( \gamma_w \) from \( p(\gamma_w) \); draw \( \gamma_I \) from \( p(\gamma_I) \);
2. suppose model A is the true model and let it generate data for one game with 8 trials, with parameter \( \gamma \). This realizes a draw from \( p(y_A|d, \gamma) \);
3. calculate the likelihood of the simulated data under model A over that under model B. This gives the Bayes factor \( BF_{A/B} \).
4. realizes a draw from \( p(y_B|d, \gamma_w, \gamma_I) \);
5. calculate \( BF_{B/A} \).
6. sum up utility = \( BF_{A/B} + BF_{B/A} \)
7. repeat 1-6 for \( J \) times, calculate the average utility associated with the current design design \( d \).
Muller Algorithm with Annealing: MCMC scheme with stationary distribution $h(d, \theta, y)$.

1. start with a design $d^0$;
2. at $d^t$, simulate $(\gamma_j^t, y_{Aj}^t)$ and $(\gamma_{wj}^t, \gamma_{lj}^t, y_{Bj}^t)$, $j = 1, \ldots, J$, for each simulated experiment evaluate $BF_{jA/B}^t$ and $BF_{jB/A}^t$;
3. evaluate $w^t = \sum_{j=1}^J \log(BF_{jA/B}^t + BF_{jB/A}^t)$;
4. propose a new candidate design $\tilde{d}^t$ from a symmetric proposal distribution (e.g. $N(d^t, \sigma^2 I)$);
5. at $\tilde{d}^t$, simulate $(\tilde{\gamma}_j^t, \tilde{y}_{Aj}^t)$ and $(\tilde{\gamma}_{wj}^t, \tilde{\gamma}_{lj}^t, \tilde{y}_{Bj}^t)$, $j = 1, \ldots, J$, for each simulated experiment evaluate $BF_{jA/B}^t$ and $BF_{jB/A}^t$;
6. evaluate $\tilde{w}^t = \sum_{j=1}^J \log(\tilde{BF}_{jA/B}^t + \tilde{BF}_{jB/A}^t)$;
7. evaluate the acceptance probability defined as $AP = min(1, e^{\tilde{w}^t - w^t})$, accept candidate with $AP$;
8. set $t = t + 1$, repeat 2-7 until convergence, all accepted $d^t$'s thereafter should represent an optimal design solution.