Sarnecka, B. W., & Carey, S. (2008). How counting represents number: What children must learn and when they learn it. *Cognition, 108*(3), 662–674.
Starkey, P., & Cooper, R. (1980). Perception of numbers by human infants. *Science, 210*, 1033–1035.
Stenning, K., & van Lambalgen, M. (2008). *Human reasoning and cognitive science*. Cambridge, MA: MIT Press.
Stich, S., & Nisbett, R. (1980). Justification and the psychology of human reasoning. *Philosophy of Science, 47*, 188–202.
Travis, C. (2008). Psychologism. In E. Lepore & B. C. Smith (Eds.), *The Oxford handbook of philosophy of language*. Oxford: Oxford University Press.
Wittgenstein, L. (1953). *Philosophical investigations* (G. E. M. Anscombe & R. Rhees, Eds., G. E. M. Anscombe, Trans.). Oxford: Blackwell.
Wittgenstein, L. (1976). *Lectures on the foundations of mathematics* (C. Diamond, Ed.). Ithaca: Cornell University Press.
Wynn, K. (1995). Origins of numerical knowledge. *Mathematical Cognition, 1*, 35–60.

2 Psychology and the A Priori Sciences

Penelope Maddy

The 'a priori sciences'[1] to be considered here are logic and arithmetic;[2] the 'psychology' includes experimental, especially developmental psychology, neurophysiology, and vision science. My goal is to examine the role these empirical theories can play in the philosophies of those sciences, or more precisely, the role I think they should play. Most of the psychological studies referred to here will be familiar to readers of this volume, though perhaps not the use to which I hope to put them.

1. Logic

Common sense tells us that much of the world comes packaged into middle-sized objects—stones, coins, snails, apples, trees, bodies of cats, apes, human beings—and not without reason; these items are what we see and touch, encounter, and engage with, in everyday life. Of course, common sense doesn't always hold up under scrutiny, but meticulous science confirms that each of these is a rough collection of molecules held together by various forces, resisting penetration due to other forces, moving as a bounded unit on a continuous spatiotemporal path.[3] Scientifically refined common sense also reveals that these objects have properties and stand in relations: stones come in a variety of sizes and shapes, apples in a various

1 I use the term 'a priori science' as the customary label, not to endorse the view that these disciplines are in fact a priori in some sense or other.
2 With a nod toward set theory in footnote 47.
3 Some philosophers question this simple view on the grounds that the commonsense table is intuitively 'solid', while the scientific object is largely empty space, so the two cannot be the same. Even assuming that common sense does picture things as continuous matter (over and above being impenetrable), it seems more natural to say science has taught us that the objects of common sense are different than we first imagined, not that they don't exist. (For a bit more on such thinking from Eddington, Sellars, and Ladyman and Ross, see Maddy 2014c, 95–97, 99, footnote 9.) Other philosophers go further, rejecting everything in science and common sense on radical skeptical grounds, but this challenge, too, I set aside for present purposes (for more, see Maddy, 2017).

colors, domestic cats are generally smaller than adult humans.[4] This simple structuring validates a certain amount of rudimentary logic: if the apple is either red or green and it's not red, then it must be green. This might all seem so obvious, so unavoidable, as to be true no matter what, true in 'all possible worlds', but in fact it breaks down at the quantum level: the particles don't behave as bounded units on continuous paths; the sense in which they enjoy properties (such as position and momentum) is problematic; some simple logical laws (e.g., the distributive law) appear to fail.[5] The inferences of this rudimentary logic are reliable as long as the requisite structure is in place, but not otherwise.

Psychology comes into this story as an investigation of how we come to those early commonsense beliefs about objects and their features. The groundbreaking developmental work of the 1980s and '90s[6] showed that young infants track cohesive, bounded, solid[7] individuals, despite occlusions, using spatiotemporal criteria such as contiguity, common fate, and continuous motion; though they're aware of object's other properties, they typically don't use regularities of shape, color, texture or motion, or features or kinds to determine object boundaries or identity.[8] Studies of neonates and nonhuman animals suggest a distant evolutionary origin.

There's some disagreement over the precise interpretation of these experimental results. The contents of 'the object concept' vary slightly from writer to writer;[9] disagreements arise over whether the abilities catalogued are

4 Dependencies between one situation and another are also important—the coin is on the floor because the cat shoved it off the table—as are universal properties, but I leave these aside for simplicity in this quick sketch of Maddy, 2007, III.4 (also (2014c)).

5 In another skeptical move, it's sometimes suggested that science can't serve to ratify the objects of common sense, because any science that begins with those objects will inevitably end up ratifying them. In fact, a science (ours) that begins with them has ended up without them in the quantum world.

6 For a summary with references, see Maddy (2007, 245–258). Carey (2009) (Chapters 2 and 3) is a much-discussed survey and philosophical elaboration by one of the leading researchers in the area. This work swept away the earlier seminal theories of Piaget (featured in Maddy (1990, 54–5)), according to which the ability to represent objects comes later in development. Carey (2009, 46–55) gives a fascinating reanalysis of Piaget's evidence.

7 That is, impenetrable (a feature of both of Eddington's tables in footnote 3).

8 Animate/inanimate, human/nonhuman appear to be exceptions. See Carey (2009, 263–284), especially pp. 276–277, for more on this point.

9 In one of the more dramatic examples, Burge departs from many psychologists (from Piaget on) in holding that the 'constitutive conditions' for representing bodies as such don't include tracking through occlusion: "A capacity to perceptually track a body as a three-dimensionally bounded and cohesive volume shape while it remains in view . . . suffices" (Burge, 2010, 460). In contrast, Hatfield (2009b) requires tracking through occlusion, but holds that the developmental evidence doesn't conclusively show infants are representing objects as "individual material objects (not as mere local collections of properties) . . . that . . . occupy . . . distinct . . . space-time worms . . . throughout their existence" Hatfield (2009, 241).

purely perceptual or somehow conceptual,[10] and so on.[11] Fortunately, these niceties needn't trouble us here because any of these options will be enough to serve as the building blocks, the 'objects with properties', in the rudimentary logical structure described earlier.

But 'object with properties' aren't all there is to that rudimentary structure: a stone has a size and a shape, an apple can be red or green, a coin can fail to be a quarter. More developmental work of the '80s and '90s shows that young infants classify cats and dogs so as to exclude birds, and even cats so as to exclude superficially similar dogs. They're also sensitive to correlations of features: infants[12] aware of three possibilities for each of the features A, B, C, D, and E (that is, A1, A2, and A3, and so on), habituated to items with correlations between these features—for example, items with $(A1 \wedge B1 \wedge C1) \wedge (D1 \vee D2) \wedge E1 \vee E2)$ and items with $(A2 \wedge B2 \wedge C2) \wedge (D1 \vee D2) \wedge (E1 \vee E2)$—find a new correlated combination (such as A2-B2-C2-D1-E2) familiar but an uncorrelated combination (such as A1-B2-C1-D1-E2) just as novel as one that's entirely new (such as A3-B3-C3-D3-E3)! In addition to these conjunctions, infants also appear sensitive to disjunctions—habituated to cats or horses, they find a dog novel—and to relations—for example, 'above', 'below', 'between'. Results like these strongly suggest that we humans are sensitive to rudimentary logical structures from an early age.[13]

Still, as is well-known, it's entirely possible to respond to a feature of the world without representing it: the frog's visual system might allow it to detect (then catch and eat) flies without representing them as flies. On this point, I'm less confident than Tyler Burge that 'representation' is a psychological natural kind[14] and even more doubtful that its contours can be

10 For example, Burge (2010, 438–450), in disagreement with Spelke (1988).

11 It could be that some of these disagreements run deeper than the sort of thing scouted in footnote 9. Sticking with Burge and Hatfield as our examples, notice that Burge takes the goal of the project to be determining what's 'constitutive' of objecthood—"Our question concerns necessary minimal constitutive conditions for having the capacity to attribute the kind *body* in perception" (Burge, 2010, 465)—where this presupposes a fact of the matter to be discovered (perhaps by rational intuition, perhaps with a hint of essentialism in the appeal to 'natural kinds'). In contrast, Hatfield (2009b, 241) only claims, "We as adult perceivers typically see (things] as individual objects" with the features listed in footnote 9 and that the developmental evidence doesn't establish that infants do this too. He describes this situation by referring to 'the adult concept', but there's no indication of an underlying Burge-like metaphysics; he could just be using the phrase to highlight the possibility of a significant cognitive shift.

12 This pattern and the next emerge in 10-month-olds, still pre-linguistic.

13 For a summary, with references, see Maddy (2007, 258–262).

14 See Burge (2010, 291): "Psychological explanations have a distinct explanatory paradigm. Psychology . . . discovers its own kinds. One of them is the kind representation".

discovered, as he suggests, by uniquely philosophical means.[15] In contrast, Gary Hatfield (1988) undertakes a more modest task, firmly grounded in contemporary vision science.[16] An ongoing debate pits those who believe that the visual system employs symbolic representations in an internal symbolic 'language'—that rules are encoded and applied, hypotheses formed and tested (in the tradition of Helmholtz)—and those who insist that the visual system is not representational, that it's simply tuned to register directly the rich and complex information available in the ever-changing array of ambient light (Gibson and his followers). Both sides acknowledge that processing takes place between the retina and the visual experience. The debate between them hinges on the question, is this processing purely physiological or does it break down into psychologically significant components and, in particular, into components with a characteristically representational role?[17] Hatfield threads the needle between the two schools of thought, arguing that there are representational components, but that they needn't involve a symbolic system.

To see how this goes, consider using a slide rule to multiply n times m: locate n on the A scale; slide 1 on the B scale beneath n on the A scale; find the number on the A scale that's above m on the B scale. The procedure works because the scales are laid out logarithmically and $n \times m = \ln^{-1}(\ln(n) + \ln(m))$. That same equation could be programmed into a digital computer and multiplication carried out in that way, in which case, the logarithmic algorithm itself would be encoded, represented, in the computer's program, but this isn't true for the slide rule: there the algorithm is effectively followed, but it isn't literally represented.[18] The lengths on the slide rule represent numbers because of what the device is designed to do (multiply, among other things) and how it was designed to do it (relying on, but not representing, the properties of logarithms). The computer is also designed to multiply (among other things), but it's designed to do so quite differently, by applying explicit rules in an internal symbolic system. So what a device does or doesn't do represent depends on how it does what it's designed to do.

15 See Burge (2010, xviii): "Philosophy has . . . a set of methodological and conceptual tools that position it uniquely to make important contributions to understanding the world. . . . Many of its topics remain of broadest human concern. Where, constitutively, representational mind begins is such a topic".

16 Obviously, this contrast (elaborated in footnotes 14 and 15) is reminiscent of the one in footnotes 9 and 11.

17 See Ullman (1980, 374) and Hatfield (1988), Section 1.

18 As Burge (2010, 504) points out, an odometer's computation of the distance traveled depends on the circumference of the tires (it records a tick for each rotation), but the circumference is nowhere represented. Hatfield (1988, 75) makes a similar point about a 'tension adder': n and m are represented by small weights placed on a pan and their sum registered by a pointer on the front of the device, but no algorithm is encoded.

If we now replace the slide rule designer or the computer programmer with the evolutionary pressures on our species,[19] then the representational status of some element of the visual system can be assessed in the same way: it depends on the function of the visual system in the evolved human organism, the function of that element within the visual system, and the method it uses to perform that function.[20] Hatfield (1988, 63–65) gives the example of seeing a circle at a slant rather than an ellipse. This function could be achieved by registering the retinal ellipse, registering slant information from shading, and computationally combining these two, or it might be achieved by a single registration of shading across the retinal ellipse. Obviously, it's an empirical matter which of these algorithms is actually implemented; it can be investigated by psychological experiments with carefully timed disruptions or by physiological investigation of the neuroanatomy. In these ways, we could determine whether or not, say, the projective retinal shape by itself is represented. But either way, it's not at all obvious that this sort of representation would involve a symbolic system.[21]

So, to return to our theme, we know that infants respond to conjunctions, disjunctions, negations, and so on, but do they actually represent them as such? As Burge notes (2010, 406), the fact that we infer in accord with a logical rule doesn't imply that the rule is somehow encoded in our psychology, presumably in some language of thought. Our concern here, though, isn't with inference, but with simple logical structuring, and (following Hatfield) the representation needn't be symbolic. The question is whether the infant represents the stone as small and round, the apple as red or green, the coin as not a quarter. Assuming that sensory sensitivity to these worldly features is adaptive,[22] a Hatfield-style answer to this question hinges on how that sensory sensitivity is achieved: does the scientific story of that ability break down into psychologically significant parts, into representational components, such as the separate representations of projective shape and

19 In practice, determining what aspects of the visual system are adaptations and which are spandrels is a very difficult undertaking. See, e.g., Warren (2012) and Anderson (2015).

20 Burge (2010), Chapter 8, soundly rejects accounts of representation based in biological function for reasons I don't fully understand and won't attempt to explicate.

21 Does this mean that the frog is representing flies? Opponents of biological function views suggest that evolutionary considerations aren't enough to show that the frog is representing flies as opposed to moving black dots, or even flies as opposed to nothing at all given that a frog's detector will occasionally go off on its own. Regarding the first point as subject to further investigation, granting it now only for the sake of argument, Hatfield contends that, nevertheless, biological function "can serve as the basis for ascribing to states of the frog's visual system the content *target fly/moving dot*, or some such coarse-grained content", that "[a]mong the functions of the frog's visual system is to represent small moving things as being there when they are, and not to represent them as being there when they aren't" (Hatfield, 1991, 122–123). In other words, the biological function account has room to regard the frog's detector as having misfired when it goes off on its own.

22 See footnote 19.

of shading information in the multistage algorithm for seeing the circle at a slant? For our case, given that the infant can represent stones, smallness, and roundness, is her representation of a small, round stone related to other representations in a way that merits describing it as a conjunctive representation? This needn't involve encoding in some language of thought any more than the circle-at-a-slant case does, but it is a straightforward empirical question for experimental psychology and neuroscience.

If a definitive answer to this question is known, it isn't known to me, but the study of visual working memory offers a hint of how a small part of it might go. Evidence suggests that we're able to store information about a limited number of objects (around four) and their features over short periods of time. This raises the question of how several features of one object are bound together: what distinguishes a scene with a vertical red bar and a horizontal green bar from one with a vertical green bar and a horizontal red one? One proposal is synchronized neural firing: a particular neuron fires repeatedly to encode a single feature; when the repeated firings of the neurons for two separate features are synchronized, they form a unit: 'cell assembly'.[23] In Hatfield's terms, the initial firings represent red, green, horizontal, and vertical bars; when the 'vertical' and 'red' neurons fire in unison, the resulting assembly represents a vertical red bar. The position I'm proposing, on pure speculation, requires that this isn't exceptional, that rudimentary logical structuring is widely represented, one way or another.

If all this is granted, what role is psychology playing in this philosophy of logic? The ground of logical truth, what makes it true (where it is true), is the objective logical structuring in the world, so there's no trace of psychologism. Psychology's role, then, might be thought to be epistemological. For example, a sufficiently externalist epistemologist, one who thinks the evolutionary pressures responsible for our logical cognition produce a reliable process,[24] might conclude that we know (at least some of) the world's logical structure a priori. I prefer to leave the policing of 'know' and 'a priori' to the specialists and to say only this much: we come to believe what we do about the logical structures in the world on the basis of primitive cognitive mechanisms, many of which we share with other animals, but our evidence for the correctness of those beliefs comes from common sense and its subsequent (partial) ratification by scientific means.

So far, this is a fairly slight philosophical impact for psychology, but I think there's an important moral concerning our philosophical preconceptions

23 See Vogel, Woodman, and Luck (2001) for discussion and references. Also Olson and Jiang (2002).

24 This needn't be a fallacious argument of the form 'this evolved, therefore it's reliable'. Instead, it might run roughly along the lines traced here: first science establishes that much of the world is logically structured; then psychology defends an evolutionary story of how we come to detect and represent that structure.

about logical truth. Because our logical beliefs rest on such primitive cognitive mechanisms, it's hard for us to see how things could be otherwise, how a world failing to instantiate those rudimentary logical forms is even possible. When quantum mechanics shows us not only that a world can fail to do this but also that our very own micro-world so fails, often the result is that we find quantum mechanics deeply problematic, not that we take logic itself to be contingent.[25] It seems to me that the psychology here is showing us *why* we're so easily inclined to believe that logical truth is necessary, a priori, certain—a stubborn preconception that vastly distorts our theorizing about it.[26] It's hard to imagine a more valuable lesson for the philosophy of logic!

2. Arithmetic

Obviously, any patch of the world with logical structuring into objects with properties, standing in relations, will also have number properties: so many objects, so many with this particular feature, so many standing in this relation to this particular individual, and so on. When it comes to our cognitive access to those number properties, though, it's well-known that the first four or so have special status: infants' expectations about how many objects will appear behind a screen after individual objects have been added or removed are accurate up to three; adults can hold three to four objects in working memory[27] and track three to four objects through complex motions, but these abilities break down quickly for higher quantities. Nonhuman animals share these abilities and limitations, indicating another primitive cognitive system.[28]

The mechanism underlying these abilities—the object-tracking or parallel individuation system—apparently includes so-called object files of mid-level vision,[29] which follow objects spatiotemporally and encode features as they go ("it's a

25 I suspect many of us have heard our fellow philosophers assert with great confidence that quantum mechanics must be false on a priori grounds.

26 See Maddy (2014a), Chapter 6, for a comparison of this conclusion with the late Wittgenstein's take on logic.

27 Feigenson (2011) describes how visual working memory can encode more than four slots worth of information by 'chunking', as when we remember a phone number by dividing it into three blocks of digits, or one of her infant subjects remembers two cats and two cars, but not four individual cats. (See also Carey (2009, 149–150).) The 'chunk' is often referred to as a 'set', exhibiting the higher ranks that differentiate sets from mere aggregates. I once appealed to analogous considerations (e.g., in Maddy (1990, 165), but for what it's worth, I'm no longer convinced anything essentially 'higher order' is involved in such cases. Seeing two cats and two cars could just be a particular way of seeing the cats and the cars, not a way of seeing something else (a set of cats, a set of cars).

28 See Maddy (2007, 319–326), for more on the story in this and the following three paragraphs, with references. See Carey (2009), Chapter 8, for her elaboration.

29 Kahneman et al. (1992). See Maddy (2007, 255–257, 319–320) for a brief discussion with references.

bird, it's a plane, it's Superman"), and visual working memory, which keeps visual information accessible over short periods.[30] The two are closely intertwined, with some evidence of complementary emphasis on tracking over motion and retention of object properties, respectively.[31] Though the infant expectation experiments are often described in arithmetic terms—1 + 1 = 2, 3 – 2 = 1—it's widely agreed that these representations are not truly numerical: not '3', but the simply logical 'a thing, another thing, and yet another thing', most likely the opening of three successive object files.[32] In cases of 'subitizing'—immediate recognition (without counting) of up to three or four objects—perhaps visual working memory is engaged, but again, mostly likely through the opening of three distinct information slots[33] rather than an explicit numerical representation.

Yet another primitive system we share with other animals is sensitive to approximate quantities: it can distinguish one dot from three more easily than two dots from three (the 'distance effect'); it can distinguish two dots from three, but not eight dots from nine (the 'magnitude effect').[34] The mechanism for this is so far unknown (at least to me), but neurological studies on monkeys suggest a two-step process that begins with a group of neurons that encode locations of objects, ignoring other features, and then feeds into an array of neurons whose responses are bell-shaped curves, each peaking at a certain number.[35] This model would explain the distance effect—the ranges of firing for 'one-neuron' and 'three-neurons' overlap less than those for 'two-neurons' and 'three-neurons'—and the magnitude effect—the bell curves for large numbers are broader. In any case, this is clearly a more quantitative system than the object tracker, but it can't truly be said to represent cardinality. Burge (2010, p. 482) suggests a return to the ancient notion of 'pure magnitude', neither continuous nor discrete, but nevertheless stands in ratios. However that may be, what matters for our purposes is that features of the world's logical structure are being represented, albeit only approximately.[36]

30 See, e.g., the references in footnote 23.

31 See Hollingworth and Rasmussen (2010). The two are often lumped together without comment, or even identified.

32 Burge points out, "There need be no use of conjunction or negation in the perceptual representation (as in: this is a body and this is a body and this is not that)" (Burge (2010, 486). He's right: it's unlikely that anything like this is encoded in a language of thought. But in Hatfield's terms, the opening of three successive object files could represent the corresponding logical feature of the scene.

33 See Chesney and Haladjian (2011) for evidence that subitizing and object tracking rely on a shared visual mechanism.

34 See, e.g., Carey (2009, 118–137) and Dehaene (2011).

35 See Dehaene (2011, 247–254) and Nieder (2011).

36 Oddly enough, on small numbers, where the two systems overlap, the infant's object-tracking system appears to override the approximate system. For example, they prefer a box where three treats have been placed to a box where one or two treats have been placed, but when the numbers are two and five, beyond the object tracker's capacity, they perform at chance—despite the fact that the ratio is big enough for the approximate system to detect easily. See Carey (2009, 84–85, 139–141, 153–155).

So far, we're in step with the nonhuman animals, still far short of human arithmetic. The leading theory is that what sets us apart is the child's ability to combine the proto-numerical fruits of the object-tracking system and the approximate system via her command of the counting sequence.[37] 'One, two, three . . .' is first learned as a verbal nonsense scheme—such as 'eeny, meeny, miney, moe . . .'—and the act of reciting it while pointing to each of a group of objects in turn is just play, of no numerical significance. Young children do realize that the use of the word 'one' correlates with the presence of a single object, with a single opened object file or a single item in visual working memory, but the sense of larger number words comes only gradually, between two and a half and three and a half: first 'two' is associated with the presence of an object and another; a few months later, 'three', and maybe even 'four', gains meaning from the object-tracking system.[38] This far the nonhuman animals can follow, but what happens next is uniquely human: apparently the child notices that an extra object in the scene corresponds to the next number in the counting sequence, and suddenly, the true meaning of counting becomes clear: the last number recited in the procedure is the number of objects in the scene.

It's sometimes assumed that this is the end of the story of how humans come to a full understanding of arithmetic, but it isn't, for at least two reasons. First, consider a child who knows how to count and knows there are 'just as many' of these as those when the same number word results from counting these as those. That is, she knows that if she counts *n* children and *n* cookies, she'll be able to give each child exactly one cookie with no cookies left over. Richard Heck makes the case that a child can know all this without having the notion of a one-to-one correspondence, which is, after all, 'very sophisticated' Heck (2000, 170). Of course, when she counts, she forms what we understand to be a one-to-one correspondence, but she needn't understand it as such; she's just implementing the counting procedure.[39] So this is one respect in which the child still hasn't grasped a notion some consider essential to the concept of 'cardinal number'.

Another tempting assumption is that a child who understands that one more object corresponds to the next number word must also understand

37 Here again, Piaget was in disagreement. See Dehaene (2011, 30–36) for an amusing account of how the empirical results were misinterpreted.

38 Some hold that the object-tracking system isn't involved, that the underlying mechanism here is the approximate number system (ANS) (see, e.g., Piazza (2011)), which is most precise for small numbers. Dehaene (2011, 256–259), who once entrusted small numbers to the ANS (what he calls 'the number sense'), explains what changed his mind.

39 Heck also notes that the child can understand 'just as many' without understanding counting: there are just as many cookies as children if she can make sure everyone has exactly one cookie with none left over. He then shows how the Peano axioms can be derived with 'just as many' in place of Frege's 'one-to-one correspondence'.

that there's no largest number.[40] The only empirical study touching on this question that I know of, Harnett and Gelman (1998), actually aims to show that it's relatively easy for children to learn that the number sequence has no end—easy compared to learning fractions!—so its design includes more coaching than would be ideal for present purposes. Still, children in kindergarten and first grade[41] did quite poorly on questions such as "is there a biggest number of all numbers?" and "is there a last number?" They did somewhat better but still far from perfectly on leading questions such as "if we count and count and count, will we ever get to the end of the numbers?" and "can we always add one more, or is there a number so big we'd have to stop?", despite having been primed with exercises in counting larger and larger numbers.[42]

Explaining their answers, the six-year-olds might suggest that we have to stop counting "'cause you need to eat breakfast and dinner" or "because we need sleep", or that we couldn't then start up again where we left off because "you forget where you stopped". There's even a hint at mortality: if we try to add one more after counting to a very big number, "I guess you'll be old, very old". Though answers like these were classified as 'unacceptable', there is a straightforward sense in which the children have it right: there *are* practical limitations on how far we're inclined to count, and even physical limits on how far we could count.[43] The young children aren't wrong exactly; they're just failing to grasp the spirit of the question. What's being asked is whether there's any limit to how far we could count, *in principle*.

In contrast to the kindergarteners and first graders, the second graders[44] in this study generally answered the questions as they were intended: there is no largest number, period. Closer analysis of the experimental results led Harnett and Gelman to the observation that the children in a position "to benefit from a conversation that offers cues" (p. 361) were those who could count beyond 100:

> Once children master the sequence from 1 to 20 and the list of decade words, they have most but not all of the vocabulary they need to apply the recursive procedures by which larger and larger numbers are generated. As they count beyond 100, they come to learn that not only the digits, but also the decade terms, are recycled over and over. [Younger] [c]hildren are still at work memorizing the teens and decade terms and are less able to appreciate that the count sequences is systematic.
>
> (Harnett & Gelman, 1998, 361)[45]

40 For a bit more on the line of thought in the remainder of this section, see Maddy (2014b).
41 Averaging just under 6 and 7 years old, respectively.
42 One group of subjects in one of the studies was questioned about the largest number, etc., before the counting exercises. Their performance was even worse than the group who did the counting exercises first.
43 Russell once remarked that running through an infinite decimal expansion is "*medically* impossible" Russell (1935/6, 143).
44 Averaging just under 8 years old.

This suggests, as the psychologist Paul Bloom proposes, that

> the generative nature of human numerical cognition develops only as a result of children acquiring the linguistic counting system of their culture. Many, but not all, human groups have invented a way of using language to talk about number, through use of a recursive symbolic grammar.
>
> (Bloom, 2000, 236)

This would mean that children's belief in the infinity of the numbers derives from their belief in the infinity of numerical expressions, not vice versa:

> [I]t is not that somehow children know that there is an infinity of numbers and infer that you can always produce a larger number word. Instead, they learn that one can always produce a larger number word and infer that there must therefore be an infinity of numbers.
>
> (Ibid., 238)

In this way, our question—how do we come to believe there's no largest number?—is pushed back one step to how do we come to believe that there's no largest numerical expression?

Harnett and Gelman's studies show that it's quite easy for children to come to this view once they've appreciated the intricacies of the systematic generation of numerical expressions. What's striking is that they don't seem bothered by concerns about the practical or physical limitations on, for example, the length of those numerical expressions or the breathe needed to utter them or the need to stop for lunch—all that apparently matters is grasping the recursive character of the rules of formation. Why is the intended 'in principle' reading of the question more natural here when it's posed for numerical expressions than it was when posed for the numbers themselves? To engage once again in rank speculation, I suggest that this traces to the recursive element of the innate linguistic faculty, whatever it is in our genetically endowed cognitive machinery that underlies our ability to understand and produce indefinitely varied and complex linguistic items:

> All approaches agree that a core property of [the linguistic faculty] is recursion . . . [The linguistic faculty] takes a finite set of elements

45 Though Harnett and Gelman speak of "recursive procedures by which larger and larger numbers are generated"; obviously, they're talking about linguistic procedures that generate numerical expressions. (Understanding that adding one results in a larger number was another predictor for successful response to the cues.)

> and yields a potentially infinite array of discreet expressions. This capacity . . . yields a discrete infinity (a property that also characterizes the natural numbers).
>
> (Hauser, Chomsky, & Fitch, 2002, 1571)

The suggestion is that this linguistic capacity is what produces our intuitive grasp of the 'in principle' question.

Assuming this sketch of the psychology is roughly right—a big assumption, subject to empirical test—the consequences for the philosophy of arithmetic are fundamental. Simple arithmetical claims such as $2 + 2 = 4$ and $12 < 191$ are ordinary facts about worldly logical structures (where they're present), but the subject matter of mathematical arithmetic—the standard model, what we now think of as an omega-sequence—doesn't depend on any contingent features of the actual world, which may or may not be finite. Insofar as arithmetic is 'about' anything, it's about an intuitive picture of a recursive sequence of potentially infinite extent—an intuitive picture we humans share thanks to the evolved linguistic faculty common to our species.

Now, we all tend to believe that the structure of the standard model of arithmetic, that simple omega-sequence, is coherent, unique, and determinate. But if it's really just a matter of an intuitive picture, what reason do we have to believe these things? As Wittgenstein once asked, "What if the picture began to flicker in the far distance?" (*RFM*, V.10). Our innate cognitive structuring may well give rise to these firm convictions, but if the story told here is correct, our capacity for mathematical arithmetic could be a mere spandrel, generated just by the way we evolved toward language, and even if it is an adaptation in itself, that's no guarantee of reliability.[46] Under the circumstances, we reflective beings should want more support for our faith in the cogency of an omega-sequence than just our brute inclination to believe it. I think there are facts we can appeal to, but they're hardly conclusive: our biological similarity as humans is reason to think your intuitive picture is more or less the same as mine; the apparent coherence of the picture, plus long experience of the species with mathematical arithmetic, provides some evidence for its consistency; the lack of any important independent statement comparable to the Continuum Hypothesis (CH) suggests it may be fully determinate.[47] But our sense that arithmetic is more secure than that may be an illusion—another valuable lesson from psychology!

46 See the fallacy described in footnote 24.

47 There's an analogous question for set theory, where the relevant intuitive picture—the iterative hierarchy—seems to rest on three elements: recursion (presumably based in the same cognitive faculty as the standard model of arithmetic); the combinatorial notion of an arbitrary subset, not beholden to any rule, definition, or construction (perhaps related to Heck's 'very sophisticated' one-to-one correspondence?); and Cantor's gutsy bet on the completed infinite (see Maddy (1988, I.5)). This picture isn't definitive of the field in the way the standard model is for arithmetic: it wasn't present when set theory was founded by Cantor and others, and it could be altered or replaced in the future (e.g., by the multiverse conception, though for now I'm skeptical about that (see Maddy (2017a, III)). In any case, given the added vagaries of the two additional elements, any case for cogency is correspondingly weaker: determinacy is undercut by independent statements such as the CH, and our biological similarity gives less support for uniqueness. Perhaps the apparent coherence of the conception delivers some evidence of consistency, but considerably less than in the case of arithmetic. Still, this would be a form of so-called intrinsic support distinct from the merely instrumental role described in Maddy (2011).

3. Conclusions

Though psychologists sometimes take their work to support a brand of anti-realism about mathematics—Stanislas Dehaene's influential *Number Sense*, for example, bears the subtitle *How the Mind Creates Mathematics*—in fact, their skepticism doesn't extend to the contingent logical/numerical structure I've been attributing to the world or our cognitive access to it:

> [A]rithmetic . . . draw[s] upon a store of fundamental knowledge accumulated over millions of years of evolution *in a physical world which, at the scale we live it, is . . . numerically structured.*
>
> (Dehaene in Dehaene and Brannon (2011, 187), emphasis added)

This type of straightforward realism breaks down, I've suggested, with the potential infinite, the standard model of arithmetic, where attention to the psychological facts reveals that our cognitive architecture does, in a sense, 'create' the subject matter under investigation. In addition to this positive semantic or metaphysical conclusion, empirical work in psychology also uncovers the less-than-firm underpinnings of some of our firmest philosophical preconceptions: that logic is necessary and that arithmetic is obviously cogent (coherent, unique, determinate). This valuable therapeutic helps free the philosophies of these subjects from traditional baggage and sets them on a more vital course. In these ways, psychological inquiry stands to play a central and highly beneficial role in our philosophizing about the a priori disciplines.[48]

References

Anderson, B. (2015). Can computational goals inform theories of vision? *Topics in Cognitive Science*, 7, 274–286.

Bloom, P. (2000). *How children learn the meanings of words*. Cambridge, MA: MIT Press.

Burge, T. (2010). *Origins of objectivity*. Oxford: Oxford University Press.

Carey, S. (2009). *The origin of concepts*. New York: Oxford University Press.

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Chesney, D., & Haladjian, H. (2011). Evidence for a shared mechanism used in multiple object tracking and subitizing. *Attention, Perception, and Psychophysics*, *73*, 2457–2480.

Dehaene, S. (2011). *The number sense: How the mind creates mathematics* (revised and expanded ed.). New York: Oxford University Press. (First edition, 1997).

Dehaene, S., & Brannon, E. (Eds.). (2011). *Space, time and number in the brain: Searching for the foundations of mathematical thought*. Amsterdam: Elsevier.

Feigenson, L. (2011). Objects, sets, and ensembles. In S. Dehaene & E. Brannon (Eds.), *Space, time and number in the brain: Searching for the foundations of mathematical thought* (pp. 13–22). Amsterdam: Elsevier.

Harnett, P., & Gelman, R. (1998). Early understandings of numbers: Paths or barriers to the construction of new understandings? *Learning and Instruction*, *8*, 341–374.

Hatfield, G. (1988). Representation and content in some (actual) theories of perception. In *Perception and cognition: Essays in the philosophy of psychology* (pp. 50–87). Oxford: Oxford University Press.

Hatfield, G. (1991). Representation in perception and cognition: Task analysis, psychological functions, and rule instantiation. In *Perception and cognition: Essays in the philosophy of psychology* (pp. 88–123). Oxford: Oxford University Press.

Hatfield, G. (2009a). *Perception and cognition: Essays in the philosophy of psychology*. Oxford: Oxford University Press.

Hatfield, G. (2009b). Getting objects for free (or not): The philosophy and psychology of object perception. In *Perception and cognition: Essays in the philosophy of psychology* (pp. 212–255). Oxford: Oxford University Press.

Hauser, M., Chomsky, N., & Fitch, T. (2002). The faculty of language: What is it, who has it, and how did it evolve? *Science*, *298*, 1569–1579.

Heck, R. (2000). Cardinality, counting, and equinumerosity. Reprinted in his *Frege's Theorem* (pp. 156–179). Oxford: Oxford University Press, 2011.

Hollingworth, A., & Rasmussen, I. (2010). Binding objects to locations: The relationship between object files and visual working memory. *Journal of Experimental Psychology: Human Perception and Performance*, *36*, 543–564.

Kahneman, D., Treisman, A., and Gibbs, B. (1992). The reviewing of object files: Object-specific integration of information. *Cognitive Psychology*, *24*, 175–219.

Maddy, P. (1988). Believing the axioms. *Journal of Symbolic Logic*, *53*, 481–511, 736–764.

Maddy, P. (1990). *Realism in mathematics*. Oxford: Oxford University Press.

Maddy, P. (2007). *Second philosophy*. Oxford: Oxford University Press.

Maddy, P. (2011). *Defending the Axioms*. Oxford: Oxford University Press.

Maddy, P. (2014a). *The logical must: Wittgenstein on logic*. New York: Oxford University Press.

Maddy, P. (2014b). A second philosophy of arithmetic. *Review of Symbolic Logic*, 7, 222–249.

Maddy, P. (2014c). A second philosophy of logic. In P. Rush (Ed.), *The metaphysics of logic* (pp. 93–108). Cambridge: Cambridge University Press.

Maddy, P. (2017). *What do philosophers do? Skepticism and the practice of philosophy*. New York: Oxford University Press.

Maddy, P. (2017a). Set-theoretic foundations. In A. Caicedo, J. Cummings, P. Koellner, and P. Larson, (Eds.), *Foundations of Mathematics* (pp. 289–322). Providence, RI: American Mathematical Society, 2017.

Nieder, A. (2011). The neural code for number. In S. Dehaene and E. Brannon (Eds.), *Space, time and number in the brain: Searching for the foundations of mathematical thought* (pp. 103–118). Amsterdam: Elsevier.

Olson, I., & Jiang, Y. (2002). Is visual short-term memory object based? Rejection of the 'strong-object' hypothesis. *Perception and Psychophysics*, *64*, 1055–1067.

Piazza, M. (2011). Neurocognitive start-up tools for symbolic number representation. In Dehaene and Brannon (Eds.), *Space, time and number in the brain: Searching for the foundations of mathematical thought* (pp. 67–285). Amsterdam: Elsevier.

Russell, B. (1935/36). The limits of empiricism. *Proceedings of the Aristotelian Society*, *36*, 131–150.

Spelke, E. (1988). Where perceiving ends and thinking begins: The apprehension of objects in infancy. In A. Yonas (Ed.), *Perceptual development in infancy* (pp. 197–234). Hillsdale, NJ: Lawrence Erlbaum.

Ullman, S. (1980). Against direct perception. *Behavioral and Brain Sciences*, *3*, 373–381.

Vogel, E., Woodman, G., & Luck, S. (2001). Storage of features, conjunction, and objects in visual working memory. *Journal of Experimental Psychology: Human Perception and Performance*, *27*, 92–114.

Warren, W. (2012). Does this computational theory solve the right problem? Marr, Gibson, and the goal of vision. *Perception*, *41*, 1053–1060.

Wittgenstein, L. (1937/44). *Remarks on the foundations of mathematics* (revised ed., G. Anscombe, Trans, G. von Wright, R. Rhees, & G. Anscombe, Eds.). Cambridge, MA: MIT Press, 1978.