The Scarring Effect of Recessions: Appendix

1 Proof of Proposition 1

According to the condition of competitive pricing and the definition of a steady state,

$$D = P_t A_t \cdot \left\{ \sum_{a=0}^{\overline{a_u}} \left[\theta_u f(\theta_u, a)(1+\gamma)^{-a} \right] + \sum_{a=0}^{\overline{a_g}} \left[\theta_g f(\theta_g, a)(1+\gamma)^{-a} \right] \right\} \tag{A1}$$

with D as the time-invariant demand, $f(\theta^e, a)$ the time-invariant number of firms with (θ^e, a), and $\overline{a_g}$, $\overline{a}_u$ the time-invariant exit ages for good and unsure firms. It suggests that $P_t A_t$ must also be time-invariant. We let $P_t A_t = PA$.

Proof. $f(0)$ represents the time-invariant entry size at the steady state. Let $V(\theta^e, a)$ be the time-invariant expected value of staying of a firm with belief θ^e and age a. The exit condition for good firms, $V(\theta_g, \overline{a_g}) = 0$, suggests:

$$\theta_g PA (1+\gamma)^{-\overline{a_g}} - 1 = 0. \tag{A2}$$

With $f(\theta^e, a)$ given by all-or-nothing learning, (A1) and (A2) together with the steady-state structure as shown in Figure 2, imply

$$f(0)\frac{(1+\gamma)^{\overline{a}_g}}{\theta_g} \left[\begin{array}{c} (\theta_u - \varphi\theta_g) \sum_{a=0}^{\overline{a}_u} \left(\frac{1-p}{1+\gamma}\right)^a + \varphi\theta_g \sum_{a=0}^{\overline{a}_g} \left(\frac{1}{1+\gamma}\right)^a + \\ \varphi\theta_g (1-p)^{\overline{a}_u+1} \sum_{a=\overline{a}_u+1}^{\overline{a}_g} \left(\frac{1}{1+\gamma}\right)^a \end{array} \right] = D. \tag{A3}$$

The free entry condition, $V(\theta_u, 0) = c_0 + c_1 f(0)$, suggests

$$\sum_{a=0}^{\overline{a}_u} \beta^a \left[\frac{PA\theta_u}{(1+\gamma)^a} - 1 \right] \lambda(\theta_u, a) + \sum_{a=0}^{\overline{a}_g} \beta^a \left[\frac{PA\theta_g}{(1+\gamma)^a} - 1 \right] \lambda(\theta_g, a) = V(\theta_u, 0) = c_0 + c_1 f(0). \tag{A4}$$

$\lambda(\theta_u, a)$ and $\lambda(\theta_g, a)$ are the probabilities of staying in operation at age a as an unsure firm and a good firm, and are given by the all-or-nothing learning.

The exit condition for unsure firms, $V\left(\theta_u, \overline{a_u}\right) = 0$, gives:

$$\theta_u PA\left(1+\gamma\right)^{-\overline{a_u}} - 1 + \beta p\varphi \sum_{a=\overline{a_u}+1}^{\overline{a_g}} \beta^{a-\overline{a_u}-1}\left[\theta_g PA\left(1+\gamma\right)^{-a} - 1\right] = 0 \tag{A5}$$

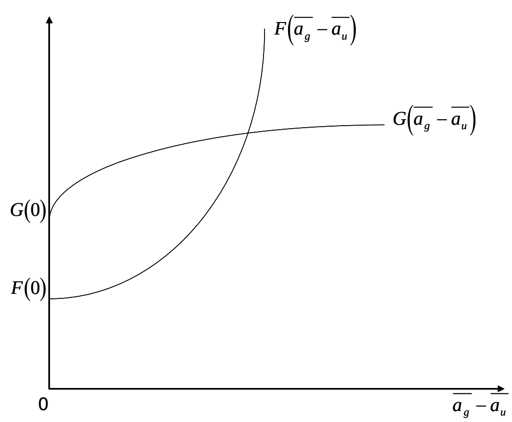
Combining (A2) and (A5) gives

$$\left(\frac{\theta_u}{\theta_g} + \frac{p\varphi\beta}{1+\gamma-\beta}\right)\left(1+\gamma\right)^{\overline{a}_g-\overline{a}_u} = 1 + \frac{p\varphi\beta}{1-\beta} - \frac{p\varphi\beta\gamma}{\left(1-\beta\right)\left(1+\gamma-\beta\right)}\beta^{\overline{a}_g-\overline{a}_u}. \tag{A6}$$

(A6) solves $\overline{a}_g - \overline{a}_u$. To establish the existence and the uniqueness of the solution, let $F\left(\overline{a}_g - \overline{a}_u\right)$ represents the left-hand side, and $G\left(\overline{a}_g - \overline{a}_u\right)$ be the right-hand side of (A6). It can be shown that $G' > 0$ but $G'' < 0$, $F' > 0$ and $F'' > 0$; moreover,

$$F\left(0\right) < G\left(0\right) \text{ as long as } \frac{\theta_u}{\theta_g} < 1$$

Since $\theta_u < \theta_g$ holds by definition, F and G must cross *once* at a positive value of $\overline{a}_g - \overline{a}_u$, as shown in the following figure



Hence, (15) determines a unique value for $\overline{a}_g - \overline{a}_u$. With $\overline{a}_u = \overline{a}_g - \left(\overline{a}_g - \overline{a}_u\right)$ and A(2), (A3) and (A4) jointly determine $f(0)$ and $\overline{a}_g$ when $c_1 = 0$.

Notice that with entry cost independent of entry size, $c_1 = 0$. (A6), (A3) and (A4) become recursive. (A6) determines $\overline{a}_g - \overline{a}_u$. With $\overline{a}_u = \overline{a}_g - \left(\overline{a}_g - \overline{a}_u\right)$, (A4) determines $\overline{a}_g$. Then (A3) determines $f(0)$. Since D is only present in (A3), variations in D would be exclusively accommodated by variations in $f(0)$. ■

2 Proof of Proposition 2

combining (A3) with (A4) and replacing $\overline{a}_u$ by $\overline{a}_g - (\overline{a}_g - \overline{a}_u)$ gives

$$\begin{aligned}
&\frac{(1+\gamma)^{\overline{a}_g}}{\theta_g}\left[\begin{array}{c} (\theta_u - \varphi\theta_g)\sum_{a=1}^{\overline{a}_u}\left(\frac{1-p}{1+\gamma}\right)^a + \varphi\theta_g\sum_{a=1}^{\overline{a}_g}\left(\frac{1}{1+\gamma}\right)^a + \\ \varphi\theta_g(1-p)^{\overline{a}_u+1}\sum_{a=\overline{a}_u+1}^{\overline{a}_g}\left(\frac{1}{1+\gamma}\right)^a \end{array}\right]\times \\
&c^{-1}\left(\frac{(1+\gamma)^{\overline{a}_g}}{\theta_g}\left\{\begin{array}{c} \sum_{a=1}^{\overline{a}_u}\beta^a\left[\begin{array}{c} (1-p)^a\left(\frac{\theta_u}{(1+\gamma)^a}-1\right)+ \\ \varphi\left(1-(1-p)^a\right)\left(\frac{\theta_g}{(1+\gamma)^a}-1\right) \end{array}\right]+ \\ \varphi\left(1-(1-p)^{\overline{a}_u+1}\right)\sum_{a=\overline{a}_u+1}^{\overline{a}_g}\beta^a\left(\frac{\theta_g}{(1+\gamma)^a}-1\right)+ \\ \theta_u - 1 \end{array}\right\}\right) \\
= \;& D
\end{aligned}$$

The left-hand monotonically increases in $\overline{a}_g$. Hence, $\frac{d(\overline{a}_g)}{dD} \geq 0$. With $\overline{a}_g - \overline{a}_u$ independent of D as suggested by (A6), $\frac{d(\overline{a}_u)}{dD} = \frac{d(\overline{a}_g - (\overline{a}_g - \overline{a}_u))}{dD} \geq 0$.

Proof. Similarly, with $\overline{a}_g - \overline{a}_u$ independent of D, $\frac{d(jd^{ss})}{dD} = \frac{d(jd^{ss})}{d\overline{a}_u}\frac{d\overline{a}_u}{dD} \leq 0$. ■

3 Proof of Proposition 3

Proof.

$$\begin{aligned}
l_g^{ss} &= \frac{\sum_{a=0}^{\overline{a}_g}\left[f(\theta_g, a) + \varphi f(\theta_u, a)\right]}{\sum_{a=0}^{\overline{a}_g}\left[f(\theta_g, a) + f(\theta_u, a)\right]} \\
&= \frac{\varphi(\overline{a}_u + 1) + \varphi\left[1-(1-p)^{\overline{a}_u+1}\right](\overline{a}_g - \overline{a}_u)}{\sum_{a=0}^{\overline{a}_u}\left[\varphi + (1-\varphi)(1-p)^a\right] + \varphi\left[1-(1-p)^{\overline{a}_u+1}\right](\overline{a}_g - \overline{a}_u)} \\
&= 1 - \frac{\sum_{a=0}^{\overline{a}_u}(1-\varphi)(1-p)^a}{\sum_{a=0}^{\overline{a}_u}\left[\varphi + (1-\varphi)(1-p)^a\right] + \varphi\left[1-(1-p)^{\overline{a}_u+1}\right](\overline{a}_g - \overline{a}_u)} \\
&= 1 - \frac{(1-\varphi)}{\frac{p\varphi(\overline{a_u}+1)}{1-(1-p)^{\overline{a_u}+1}} + (1-\varphi) + p\varphi\left(\overline{a_g} - \overline{a_u}\right)}
\end{aligned}$$

(15) implies that $\overline{a}_g - \overline{a_u}$ is independent of D, so that

$$\frac{d\left(l_g\right)}{d\left(D\right)} = \frac{d\left(r_g\right)}{d\left(\overline{a_u}\right)} \frac{d\left(\overline{a_u}\right)}{d\left(D\right)}$$

Proposition 3 has established that $\frac{d(\overline{a_u})}{d(D)} \geq 0$. Therefore, $\frac{d(l_g)}{d(D)} \geq 0$ if and only if $\frac{d(l_g)}{d(\overline{a_u})} \geq 0$. With $\frac{\overline{a_u}+1}{1-(1-p)^{\overline{a_u}+1}} = x$, $\frac{d(l_g)}{d(\overline{a_u})} = \frac{d(l_g)}{d(x)} \frac{d(x)}{d(\overline{a_u})}$. Since $\frac{d(l_g)}{d(x)} > 0$, $\frac{d(l_g)}{d(\overline{a_u})} \geq 0$ if and only if $\frac{d(x)}{d(\overline{a_u})} \geq 0$. Hence, we need to prove that $\frac{d(x)}{d(\overline{a_u})} \geq 0$.

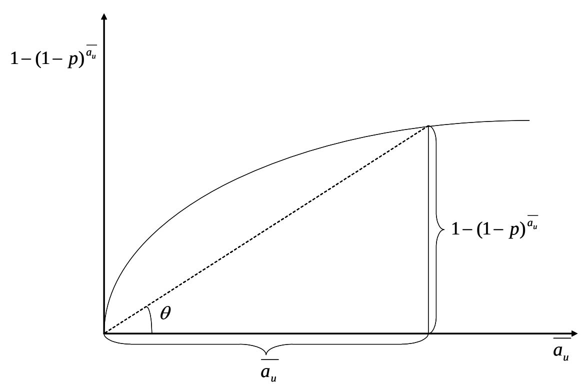
$1-(1-p)^{\overline{a_u}+1}$ is plotted in the following graph as a function of $\overline{a_u}+1$. Since

$$\frac{d\left(1-(1-p)^{\overline{a_u}+1}\right)}{d\left(\overline{a_u}+1\right)} = -\left(1-p\right)^{\overline{a_u}+1} \ln\left(1-p\right) > 0$$

but

$$\frac{d^2\left(1-(1-p)^{\overline{a_u}+1}\right)}{d\left(\overline{a_u}+1\right)^2} = -\left(1-p\right)^{\overline{a_u}+1} \left(\ln\left(1-p\right)\right)^2 < 0,$$

the curve is concave.



Clearly, it indicates that $x = \frac{\overline{a_u}+1}{1-(1-p)^{\overline{a_u}+1}} = \cot\left(\theta\right)$.The concavity of the curve suggests that as $\overline{a_u}$ increases, the angle of θ shrinks and $\cot\left(\theta\right)$ increases. Therefore, x increases in $\overline{a_u}$. ■

4 Calibrating High Demand, Low Demand, and the Entry Cost

The algorithm includes the following steps:

1. Loop around three conditions to find $\overline{a_g}h$. It generates an average firm age of around 58 quarters, a mean entry rate of around 3.11%, and a mean exit rate of around 3.44%.

2. Let the exit margin to shift from $\overline{a_g}h$ to younger age quarter by quarter, until it generates the observed peak in exit rate. We then use the observed trough in entry rate to find the *proportional* drop in entry size at this moment. This is when low demand hits the high-demand equilibrium.

3. Use the size of the shift and the proportional drop in entry size to calculate the output at this moment normalized by the high-demand entry size, which, combined with (A2) from Appendix 1, gives us $\frac{Dl}{fh}$.

4. Assume that demand stays low. Let the exit margin to move to older age quarter by quarter; meanwhile, calculate the exit rate arising from the learning margin. Stops when it reaches the observed minimum exit rate. Move the exit margin back by one quarter – this is where $\overline{a_g}l$ positions.

5. Calculate $\frac{Dl}{fl}$ using (A3), which, together with $\frac{Dl}{fh}$ from Step 3, determines $\frac{fh}{fl}$.

6. Use (A2) and (A4) to find the entry values: Vh and Vl. Calculate c_0 by equating $\frac{fh}{fl}$ to $\frac{Vh-c_0}{Vl-c_0}$.

7. Calculate fh and fl using $Vh = c_0 + c_1 fh$ and $Vl = c_0 + c_1 fl$.

8. With fh and fl, we calculate Dh and Dl using (A3).

5 Approximating Value Functions with the Krusell-Smith Approach

The key computational task is to map F, the firm distribution across ages and idiosyncratic productivity, given demand level D, into a set of value functions $V(\theta^e, a; F, D)$. To make the state space tractable, we define a variable X such that:

$$X(F) = \sum_a \sum_{\theta^e} (1+\gamma)^{-a} \theta^e f(\theta^e, a). \tag{A7}$$

Combining (A7) with (8) and (9) in the article gives

$$P(F, D) A = \frac{D}{X(F')}.$$

A is the leading technology; F' is the updated firm distribution after entry and exit; X' corresponds to F'; $P(F, D)$ is the equilibrium price in a period with initial aggregate state

(F, D). Since $F' = H(F, D)$, the above equation can be re-written as

$$P(F, D) A = \frac{D}{X(H(F, D))}$$

Given these definitions, the single-period profitability of a firm of idiosyncratic productivity θ^e and age a, given aggregate state (F, D), equals

$$\pi(a, \theta; F, D) = \frac{D}{X(H(F, D))}(1 + \gamma)^{-a}(\theta + \varepsilon) - 1. \tag{A8}$$

Thus, the aggregate state (F, D) and its law of motion help firms to predict future profitability by suggesting sequences of X's from today onward under different paths of demand realizations. The question then is: what is the firm's critical level of knowledge of F that allows it to predict the sequence of $X's$ over time? Although firms would ideally have full information about F, this is not computationally feasible. Therefore we need to find an information set Ω that delivers a good approximation of firms' equilibrium behavior, yet is small enough to reduce the computational difficulty.

I look for an Ω through the following procedure. In step 1, we choose a candidate Ω. In step 2, we postulate perceived laws of motion for all members of Ω, denoted H_Ω, such that $\Omega' = H_\Omega(\Omega, D)$. In step 3, given H_Ω, we calculate firms' value functions on a grid of points in the state space of Ω applying value function iteration, and obtain the corresponding industry-level decision rules – entry sizes and exit ages across aggregate states. In step 4, given such decision rules and an initial firm distribution. We simulate the behavior of a continuum of firms along a random path of demand realizations, and derive the implied aggregate behavior — a time series of Ω. In step 5, we use the stationary region of the simulated series to estimate the *implied* laws of motion and compare them with the *perceived* H_Ω; if different, we update H_Ω, return to step 3 and continue until convergence. In step 6, once H_Ω converges, we evaluate the fit of H_Ω in terms of tracking the aggregate behavior. If the fit is satisfactory, we stop; if not, we return to step 1, make firms more knowledgeable by expanding Ω, and repeat the procedure.

I start with $\Omega = \{X\}$ — firms observe X instead of F. We further assume that firms perceive the sequence of future coming $X's$ as depending on nothing more than the current observed X and the state of demand. The perceived law of motion for X is denoted H_x so that $X' = H_x(X, D)$. We then apply the procedure described above and simulate the behavior of a continuum of firms over 5000 periods. The results are presented in Table 1.

The estimated H_x is log-linear. The fit of H_x is quite good, as suggested by the high R^2, the low standard forecast error, and the low maximum forecast error. The good fit when $\Omega = \{X\}$ implies that firms perceiving these simple laws of motion make only small mistakes in forecasting future prices. To explore the extent to which the forecast error can be explained by variables other than X, we implement the Den Haan and Marcet (1994) test using instruments $[1, X, \mu_a, \sigma_a, \gamma_a, \kappa_a, r_u]$, where μ_a, σ_a, γ_a, κ_a,r_u are the mean, standard deviation, skewness, and kurtosis of the age distribution of firms, and the fraction of unsure firms, respectively. The test statistic is 0.7343, well below the critical value at the 1% level. This suggests that given the estimated laws of motion, we do not find much additional forecasting power contained in other variables.

Figure A1 displays the value of staying for heterogeneous firms as a function of a, θ^e, D and X ($\log X$). Figure A2 displays the corresponding optimal exit ages and entry sizes. These tables and figures suggest that our solution using X to approximate the aggregate state closely replicates optimal firm behavior at the equilibrium. These results were robust when experimented with different parameterization of the model. Therefore, we use the solution based on $\Omega = \{X\}$ to generate all the relevant series.

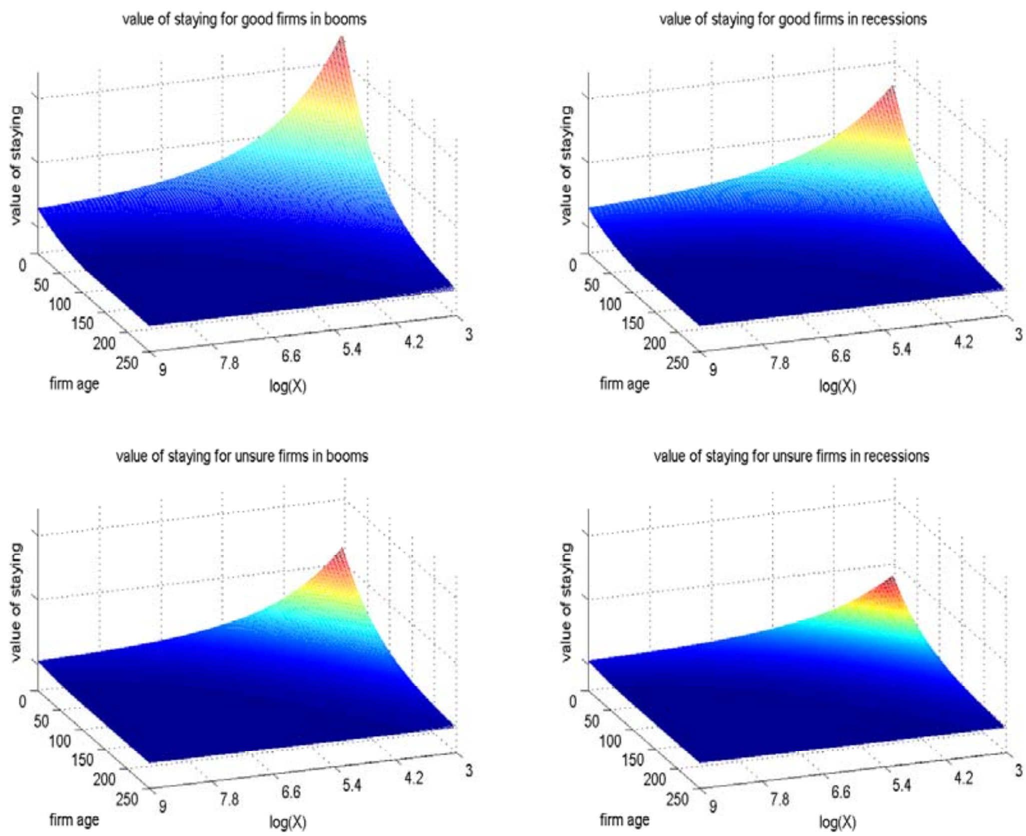


Figure 1: Figure A1: Expected Value of Staying: aggregate state variables are D and $\log X$ (the log of detrended output), firm-level state variables are firm age and expected idiosyncratic productivity (good or unsure); the parameter choices underlying these figures are summarized in Table 1 in the article.

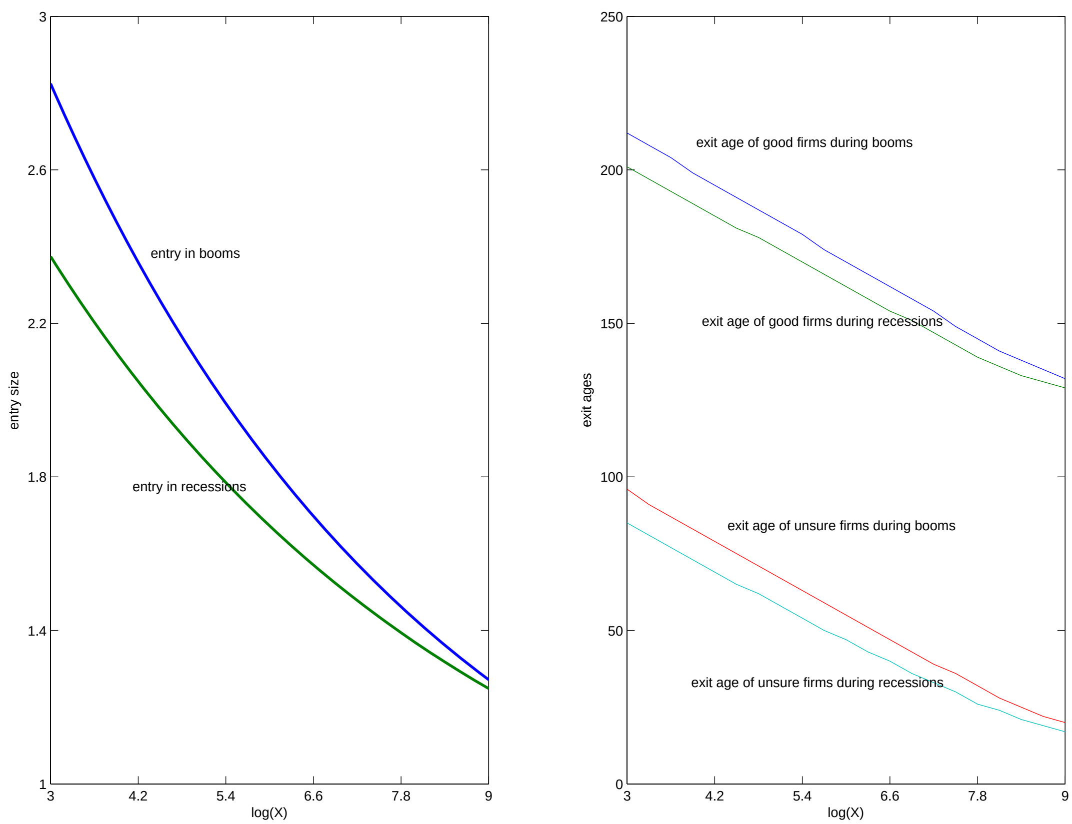


Figure 2: Figure A2: Industry-level Policy Functions: Entry Size and Exit Ages. Aggregate states are D (booms or recessions) and $\log X$ (the log of detrended output).

Ω	$\{X\}$
H_Ω	$H_x(X, D_h)$: $\log X' = 0.7565 + 0.9118 \log X$ $H_x(X, D_l)$: $\log X' = 1.9672 + 0.7647 \log X$
R^2	for D_h: 0.9994 for D_l: 0.8747
standard forecast error	for D_h: 0.0000029724% for D_l: 0.000032543%
maximum forecast error	for D_h: 0.0000627% for D_l: 0.0008125%
Den Haan & Marcet test statistic (χ^2_7)	0.9536

Table 1: The Estimated Laws of Motion and Measures of Fit