Criteria for color constancy in trichromatic bilinear models

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We examine conditions under which the spectral properties of lights and surfaces may be recovered by a trichromatic visual system that uses bilinear models. We derive criteria for perfect recovery, formulated in terms of invariant properties of model matrices, for situations in which either two or three lights are shone sequentially on a set of surfaces.

Key words: color constancy, spectral recovery

INTRODUCTION

In recent research[1-3] we studied color-constancy algorithms that use the lights reflected from surfaces seen under changing illumination. We derived a general, two-stage linear recovery algorithm for recovering spectral descriptions of lights and surfaces from quantum catches. The algorithm uses bilinear models of how quantum catches depend on illuminant spectral power distributions, surface-reflectance functions, and visual-system photoreceptoral spectral sensitivities.[1-4]

We investigated conditions under which the recovery algorithm works flawlessly.[2,3] We showed that the algorithm can fail even when the number of quantum-catch data compares favorably with the number of unknown spectral descriptors. For example, a dichromatic visual system cannot recover two-dimensional descriptions of lights and surfaces when provided reflected lights from two surfaces.

For a recovery algorithm to work perfectly, it is necessary and sufficient that the algorithm's bilinear model provide a one-to-one relationship between sets of lit surfaces and quantum-catch data. If different sets of lit surfaces give rise to identical quantum-catch data, there is no hope of recovering unique spectral descriptions of lights and surfaces from quantum-catch data.[2,3]

For two-stage linear recovery procedures, the necessary and sufficient conditions for perfect recovery can be expressed in terms of a homogeneous system of polynomial equations that has only the trivial zero solution if recovery is to work. These equations form the basis of a model-check algorithm for proving color-constancy theorems.[2,3] Although the model-check algorithm is numerically tractable, its function is not overly accessible to intuition.

We present here a more intuitive criterion for the perfect function of trichromatic color-constancy schemes. We show how invariant properties of bilinear model matrices afford simple criteria for the perfect recovery of three spectral descriptors per surface reflectance by a trichromatic visual system. Bilinear model matrices must differ in some fundamental way from one another if recovery is to be possible, for if they do not, different sets of lit surfaces would give rise to identical quantum-catch data and so be indistinguishable. The invariant properties of irreducibility and indecomposability capture this intuition and provide our criteria. The methods that we use complement our earlier eigenanalysis[2,3] of the recovery of two spectral descriptors per surface reflectance. This research was presented in preliminary form elsewhere.[5]

TRICHROMATIC BILINEAR MODELS

We use the notation of D'Zmura[1] and D'Zmura and Iverson.[2,3] We first construct a bilinear model of the interaction among reflectances, illuminants, and photoreceptors. A single such model instantiates particular choices of finite-dimensional linear models for reflectances and illuminants and a choice of three photoreceptoral spectral sensitivities, $Q_k(\lambda), k = 1, 2, 3$.

We take surface reflectance functions $R(\lambda)$ to fall within the span of some known, three-dimensional model for reflectances.[6,7] The model provides three orthogonal basis functions, $R_j(\lambda), j = 1, 2, 3$, that are combined linearly to match a particular reflectance:

$$R(\lambda) = \sum_{j=1}^{3} r_j R_j(\lambda),$$

where the descriptors r_j of the reflectance $R(\lambda)$ are given by

$$r_j = \int R(\lambda) R_j(\lambda) \mathrm{d}\lambda .$$

Likewise, a three-dimensional model for illumination provides three basis functions, $A_i(\lambda), i = 1, 2, 3$, that are combined to match a particular illuminant that falls within the span of the model[8,9]:

$$A(\lambda) = \sum_{i=1}^{3} a_i A_i(\lambda),$$

where the descriptors a_i of the illuminant $A(\lambda)$ are given by

$$a_i = \int A(\lambda) A_i(\lambda) \mathrm{d}\lambda .$$

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We continue by defining the three bilinear model matrices $\boldsymbol{\beta}_k$ with entries b_{jik} given by

$$b_{jik} = \int R_j(\lambda)A_i(\lambda)Q_k(\lambda)\mathrm{d}\lambda\,, \tag{1}$$

thus providing one bilinear model matrix per photoreceptoral spectral sensitivity. We find that the quantum catch q_k of the kth photoreceptoral mechanism is given by

$$q_k = \sum_{j=1}^{3}\sum_{i=1}^{3} r_j b_{jik} a_i \qquad \text{for } k = 1, 2, 3\,. \tag{2}$$

We now extend the model to treat multiple views of multiple surfaces.[1] Each view of a surface is provided by a distinct illuminant. Lighting a single surface successively with each of v illuminants provides v quantum-catch vectors, each with entries q_k of the form given in Eq. (2). Similarly, introducing s surfaces with linearly independent vectors of descriptors, each of which is lit, in turn, by the v illuminants, provides sv quantum-catch vectors. In what follows we assume that $s = 3$.

Introducing the index t to range over the three surfaces and the index w to range over the v views, we have the following generalization of Eq. (2):

$$q_{twk} = \sum_{j=1}^{3}\sum_{i=1}^{3} r_{tj} b_{jik} a_{iw} \qquad \text{for } t = 1, 2, 3\,, \quad w = 1, \ldots, v, \qquad k = 1, 2, 3\,. \tag{3}$$

We introduce the following matrices: $\boldsymbol{\Delta}_k$ with entries q_{twk}, $\mathbf{R}$ with entries r_{tj}, and $\mathbf{A}$ with entries a_{iw}. Then Eq. (3) reads

$$\Delta_k = \mathbf{R}\boldsymbol{\beta}_k\mathbf{A}, \qquad k = 1, 2, 3\,. \tag{4}$$

The matrix of reflectances $\mathbf{R}$ and the three bilinear model matrices $\boldsymbol{\beta}_k$ are 3×3, and the matrix of illuminants $\mathbf{A}$ and the data matrices $\boldsymbol{\Delta}_k$ are $3 \times v$.

CRITERIA

We first provide a criterion for the perfect recovery of three descriptors for each illuminant and surface reflectance in the situation in which three views of three surfaces are provided. This criterion provides a springboard for the second criterion, which concerns the problem of two views of three surfaces that was introduced by D'Zmura.[1]

Given data $\boldsymbol{\Delta}_k$ and matrices $\boldsymbol{\beta}_k$, recovery algorithms seek a reflectance matrix $\mathbf{R}$ and an illuminant matrix $\mathbf{A}$ that satisfy Eq. (4). Note that if some pair $(\mathbf{R}, \mathbf{A})$ satisfies Eq. (4) for some fixed (but arbitrary) data, then so will the pair $(\mathbf{R}/c, c\mathbf{A})$ for any choice of nonzero c. However, there may be other pairs of reflectances and illuminants not of the form $(\mathbf{R}/c, c\mathbf{A})$ that also satisfy Eq. (4). In this event, Eq. (4) cannot be inverted successfully: algorithms for recovering spectral descriptors, such as those of Maloney and Wandell[10] and of D'Zmura,[1] cannot work. The purpose of this paper is to describe criteria on the bilinear model matrices $\boldsymbol{\beta}_k$ that, when satisfied, guarantee that the only solutions to Eq. (4) are of the form $(\mathbf{R}/c, c\mathbf{A})$.

Suppose that given data $\boldsymbol{\Delta}_k$ can be expressed in terms of both a matrix pair $(\mathbf{R}, \mathbf{A})$ and a matrix pair $(\mathbf{S}, \mathbf{Z})$; i.e.,

$$\boldsymbol{\Delta}_k = \mathbf{R}\boldsymbol{\beta}_k\mathbf{A} = \mathbf{S}\boldsymbol{\beta}_k\mathbf{Z}, \qquad k = 1, 2, 3\,.$$

Because matrix $\mathbf{S}$ and matrices $\boldsymbol{\beta}_k$ are assumed invertible, one has

$$\boldsymbol{\beta}_k^{-1}\mathbf{S}^{-1}\mathbf{R}\boldsymbol{\beta}_k\mathbf{A} = \mathbf{Z}, \qquad k = 1, 2, 3\,.$$

With $\mathbf{E} = \mathbf{S}^{-1}\mathbf{R}$, we have, equivalently,

$$(\boldsymbol{\beta}_k^{-1}\mathbf{E}\boldsymbol{\beta}_k - \boldsymbol{\beta}_1^{-1}\mathbf{E}\boldsymbol{\beta}_1)\mathbf{A} = \mathbf{0}, \qquad k = 2, 3\,. \tag{5}$$

At this point it is useful to introduce the associated model matrices $\mathbf{G}_{k1} = \boldsymbol{\beta}_k\boldsymbol{\beta}_1^{-1}$, $k = 2, 3$. In terms of these matrices, Eq. (5) becomes

$$[\mathbf{G}_{k1}, \mathbf{E}]\boldsymbol{\beta}_1\mathbf{A} = \mathbf{0}, \qquad k = 2, 3\,, \tag{6}$$

where we have used standard notation $[\mathbf{X}, \mathbf{Y}]$ to denote the commutator product $\mathbf{XY} - \mathbf{YX}$ of square matrices $\mathbf{X}, \mathbf{Y}$ of common dimension. We seek criteria on the matrices $\mathbf{G}_{21}$, $\mathbf{G}_{31}$ so that regardless of the matrix $\mathbf{A}$ of illuminant descriptors, the only matrices $\mathbf{E}$ that satisfy Eq. (6) are of the form

$$\mathbf{E} = c\mathbf{I}\,, \tag{7}$$

for some nonzero constant c. Note that if $\mathbf{E} = c\mathbf{I}$ are the only solutions of Eq. (6), then the pairs $(\mathbf{R}/c, c\mathbf{A})$ are the only solutions of Eq. (4); i.e., recovery is unique.

To facilitate compact expression of the desired criteria, we now introduce appropriate terminology.

Irreducible, Indecomposable, and Regular Models

A square matrix $\mathbf{X}$ acting in a vector space W has an invariant subspace V if $\mathbf{Xv} \in V$ for all $\mathbf{v} \in V$. If V is nontrivial, i.e., $V \neq \{0\}$ and $V \neq W$, then $\mathbf{X}$ is reducible. If $\mathbf{X}$ is reducible, so is any matrix similar to $\mathbf{X}$; i.e., if $\mathbf{Y} = \mathbf{MXM}^{-1}$ for some nonsingular matrix $\mathbf{M}$ ($\mathbf{Y}$ is similar to $\mathbf{X}$), then $\mathbf{Y}$ is reducible if and only if $\mathbf{X}$ is reducible. A matrix that is not reducible is said to be irreducible. By an appropriate similarity transformation, a reducible 3×3 matrix can be brought into one of two forms, according to whether the invariant subspace is one or two dimensional. These forms are, respectively,

$$\begin{bmatrix} x & x & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}, \qquad \begin{bmatrix} x & 0 & 0 \\ x & x & x \\ x & x & x \end{bmatrix}, \tag{8a}$$

where x denotes an entry that is not necessary zero. We define a trichromatic model to be reducible if both $\mathbf{G}_{21}$ and $\mathbf{G}_{31}$ are reducible with respect to a common invariant subspace. A model that is not reducible is irreducible.

A second critical concept is decomposability, which is a special kind of reducibility.[11] If V_1, V_2 are subspaces of a linear space W, and every vector $\mathbf{w} \in W$ can be written in the form $\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2$ with $\mathbf{v}_1 \in V_1$ and $\mathbf{v}_2 \in V_2$ in one and only one way, then W is the direct sum of V_1 and V_2, and one writes $W = V_1 \oplus V_2$. If V_1 and V_2 are nontrivial invariant subspaces for $\mathbf{X}$, and $W = V_1 \oplus V_2$, then $\mathbf{X}$ is decomposable (or fully reducible). All matrices that are similar to a decomposable matrix are themselves decomposable. By an appropriate similarity transformation, a decomposable 3×3 matrix can be brought into the form

$$\begin{bmatrix} x & 0 & 0 \\ 0 & x & x \\ 0 & x & x \end{bmatrix}. \tag{8b}$$

We define a trichromatic model to be decomposable if both $\mathbf{G}_{21}$ and $\mathbf{G}_{31}$ can be brought simultaneously into the form of expression (8b) under a similarity transformation. A model that is not decomposable is indecomposable. We show below that the notions of irreducibility and indecomposability help to guarantee that the only solutions to Eq. (4) are of the form $(c\mathbf{R}, \mathbf{A}/c)$.

Finally, we say that a trichromatic model is regular if at least one of its associated model matrices $\mathbf{G}_{21}, \mathbf{G}_{31}$, or $\mathbf{G}_{23} = \mathbf{G}_{21}\mathbf{G}_{31}{}^{-1}$ has distinct eigenvalues. All models that we have encountered in a practical context are regular, and in what follows we deal primarily with this class of models. Model regularity implies that one can always find a similarity transformation that renders one of $\mathbf{G}_{21}, \mathbf{G}_{31}$, or $\mathbf{G}_{23}$ diagonal. Note that such a transformation does not alter the structure of Eq. (6). We label the photoreceptors in such a way that it is $\mathbf{G}_{21}$ that is chosen diagonal with distinct eigenvalues:

$$\mathbf{G}_{21} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \tag{9}$$

with $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq \lambda_3$, and $\lambda_2 \neq \lambda_3$.

Three Views of Three Surfaces

Let us examine the problem in which a trichromatic visual system attempts to recover three spectral descriptors for each of three linearly independent illuminants and three independent surfaces, in the case in which the illuminants are shone sequentially on the three surfaces. Matrix $\mathbf{A}$ is then a 3×3 invertible matrix, and Eq. (6) simplifies to

$$[\mathbf{G}_{21}, \mathbf{E}] = \mathbf{0} = [\mathbf{G}_{31}, \mathbf{E}]. \tag{10}$$

That is, matrix $\mathbf{E}$ commutes with both $\mathbf{G}_{21}$ and $\mathbf{G}_{31}$. A well-known result from linear algebra, namely Schur's lemma (see Lang[12]) applies: if a trichromatic model is irreducible, then the only solutions of Eq. (10) [or, equivalently, Eq. (6)] are given by Eq. (7): recovery is unique.

This result applies to all trichromatic models, regular or not. However, within the class of regular models we can establish a stronger result:

Theorem 1. If a trichromatic model is regular and indecomposable, then the only solutions $\mathbf{E}$ of Eq. (10) are of the desired form given by Eq. (7). Conversely, if a trichromatic model is decomposable, then there always exist solutions $\mathbf{E}$ of Eq. (10) that are not of the form of Eq. (7).

Proof. (Necessity.) The model is assumed to be regular, so that $\mathbf{G}_{21}$ can be taken as diagonal [Eq. (9)]. In terms of the entries e_{ij} of $\mathbf{E}$, we find that

$$[\mathbf{G}_{21}, \mathbf{E}] = \begin{bmatrix} 0 & (\lambda_1 - \lambda_2)e_{12} & (\lambda_1 - \lambda_3)e_{13} \\ (\lambda_2 - \lambda_1)e_{21} & 0 & (\lambda_2 - \lambda_3)e_{23} \\ (\lambda_3 - \lambda_1)e_{31} & (\lambda_3 - \lambda_2)e_{32} & 0 \end{bmatrix}, \tag{11}$$

and by Eq. (10) this matrix is null. Since eigenvalues λ_1, λ_2, and λ_3 of $\mathbf{G}_{21}$ differ from one another, we obtain $e_{ij} = 0$ for $i \neq j$. In other words, $\mathbf{E}$ is diagonal. The commutator $[\mathbf{G}_{31}, \mathbf{E}]$ is then given by

$$[\mathbf{G}_{31}, \mathbf{E}] = \begin{bmatrix} 0 & (e_{22} - e_{11})g_{12} & (e_{33} - e_{11})g_{13} \\ (e_{11} - e_{22})g_{21} & 0 & (e_{33} - e_{22})g_{23} \\ (e_{11} - e_{33})g_{31} & (e_{22} - e_{33})g_{32} & 0 \end{bmatrix}, \tag{12}$$

where the g_{ij} are the entries of the matrix $\mathbf{G}_{31}$. Since $[\mathbf{G}_{31}, \mathbf{E}] = \mathbf{0}$ by Eq. (10), it follows from Eq. (12) that either

(a) $e_{11} = e_{22} = e_{33}$

or

(b1) $e_{11} \neq e_{22}$ and $e_{11} \neq e_{33}$,

or (b2) $e_{11} \neq e_{22}$ and $e_{22} \neq e_{33}$,

or (b3) $e_{11} \neq e_{33}$ and $e_{22} \neq e_{33}$.

In case (b1) we have $g_{12} = g_{21} = g_{13} = g_{31} = 0$, and $\mathbf{G}_{31}$ decomposes [it has the canonical form of expression (8b)]. Possibilities (b2) and (b3) are identical to case (b1) except for a permutation of rows and columns. Because the model is assumed indecomposable, only case (a) can occur; i.e., $\mathbf{E} = c\mathbf{I}$ for some nonzero constant c.

(Sufficiency.) Because the model is decomposable, an appropriate similarity transformation renders both $\mathbf{G}_{21}$ and $\mathbf{G}_{31}$ in the form of expression (8b). It is then easily checked that

$$\mathbf{E} = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & e_{22} & 0 \\ 0 & 0 & e_{22} \end{bmatrix}$$

commutes with both $\mathbf{G}_{21}$ and $\mathbf{G}_{31}$. For $e_{11} \neq e_{22}$ these solutions $\mathbf{E}$ of Eq. (10) [or equivalently, Eq. (6)] are not of the desired form $\mathbf{E} = c\mathbf{I}$. ∎

For the current problem, in which a trichromatic visual system is provided three independent views of three independent surfaces, the above theorem shows that the set of all regular models is partitioned as a pair of disjoint classes: the class of indecomposable models, which provide perfect recovery of illuminant and surface-reflectance descriptors, and the class of decomposable models, which are "total failures"[2], i.e., models for which unique recovery of illuminant and reflectance descriptors is never possible.

Two Views of Three Surfaces

For this problem the two columns of $\boldsymbol{\beta}_1\mathbf{A}$ span a two-dimensional subspace of $\Re^3$. Let $\boldsymbol{\alpha}$ be perpendicular to this subspace, i.e., $\boldsymbol{\alpha}^T\boldsymbol{\beta}_1\mathbf{A} = \mathbf{0}$. Then Eq. (6) is equivalent to the pair of equations

$$[\mathbf{G}_{21}, \mathbf{E}] = \mathbf{z}\boldsymbol{\alpha}^T, \tag{13a}$$

$$[\mathbf{G}_{31}, \mathbf{E}] = \mathbf{z}^*\boldsymbol{\alpha}^T, \tag{13b}$$

where $\mathbf{z}, \mathbf{z}^*$ are 3-vectors. For this problem, the analog of Theorem 1 reads as follows:

Theorem 2. If a regular trichromatic model is irreducible, then the only solutions $\mathbf{E}$ of Eqs. (13a) and (13b) are of the form given by Eq. (7). Conversely, if a regular trichromatic model is reducible, there always exist solutions $\mathbf{E}$ of Eqs. (13a) and (13b) [or, equivalently, Eq. (6)], which do not conform to Eq. (7).

Proof. (Necessity.) By regularity, we may assume, without loss of generality, that $\mathbf{G}_{21}$ is diagonal and is given by Eq. (9). The commutator $[\mathbf{G}_{21}, \mathbf{E}]$ is as given earlier in Eq. (11).

We proceed by specializing the vector $\boldsymbol{\alpha}$ that specifies, in part, the right-hand terms in Eqs. (13a) and (13b). There are three cases to consider: (a) no element of $\boldsymbol{\alpha}$ is zero, (b) exactly one element of $\boldsymbol{\alpha}$ is zero, and (c) exactly two elements of $\boldsymbol{\alpha}$ are zero.

Case (a). $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 \neq 0$. Writing out the right-hand term of Eq. (13a) in detail, we have

$$[\mathbf{G}_{21}, \mathbf{E}] = \begin{bmatrix} z_1\alpha_1 & z_1\alpha_2 & z_1\alpha_3 \\ z_2\alpha_1 & z_2\alpha_2 & z_2\alpha_3 \\ z_3\alpha_1 & z_3\alpha_2 & z_3\alpha_3 \end{bmatrix}, \tag{14}$$

and from Eq. (11) we see that the diagonal terms $z_i\alpha_i$ vanish for each $i = 1, 2, 3$. Since $\alpha_i \neq 0, i = 1, 2, 3$, we have $\mathbf{z} = \mathbf{0}$. It follows that $[\mathbf{G}_{21}, \mathbf{E}] = \mathbf{0}$, and by Eq. (11) we have $e_{ij} = 0, i \neq j$. Thus $\mathbf{E}$ is diagonal, and the commutator $[\mathbf{G}_{31}, \mathbf{E}]$ is given by both Eq. (12) and Eq. (13b). When written out in detail, Eq. (13b) takes the form of Eq. (14) with z_i replaced by z_i^*. Comparing diagonal terms in Eqs. (12) and (13b), we deduce that $z_i^* = 0$ for $i = 1, 2, 3$. Thus $[\mathbf{G}_{31}, \mathbf{E}] = \mathbf{0}$. By assumption, the model is irreducible, and *a fortiori*, indecomposable. Theorem 1 applies and assures us that the only solutions of Eqs. (13a) and (13b) are those of the form $\mathbf{E} = c\mathbf{I}$ for some $c \neq 0$. (We could also apply Schur's lemma to reach the same conclusion.)

Case (b). $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\alpha_3 = 0$. (The possibilities $\alpha_1 = 0, \alpha_2 \neq 0, \alpha_3 \neq 0$ and $\alpha_1 \neq 0, \alpha_2 = 0, \alpha_3 \neq 0$ are simply notational variants of the case that we treat in detail.) Comparison of Eqs. (11) and (14) reveals that $z_1 = z_2 = 0$ and $e_{12} = e_{21} = e_{13} = e_{23} = 0$. That is,

$$\mathbf{E} = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & e_{22} & 0 \\ e_{31} & e_{32} & e_{33} \end{bmatrix},$$

$$[\mathbf{G}_{21}, \mathbf{E}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z_3\alpha_1 & z_3\alpha_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (\lambda_3 - \lambda_1)e_{31} & (\lambda_3 - \lambda_2)e_{32} & 0 \end{bmatrix}. \tag{15}$$

If $z_3 = 0$, we obtain $e_{31} = 0$ and $e_{32} = 0$, and $\mathbf{E}$ is diagonal. The commutator $[\mathbf{G}_{31}, \mathbf{E}]$ is then given by both Eq. (12) and the following expression, which follows from Eq. (13b) and from the fact that α_3 is zero:

$$[\mathbf{G}_{31}, \mathbf{E}] = \begin{bmatrix} z_1^*\alpha_1 & z_1^*\alpha_2 & 0 \\ z_2^*\alpha_1 & z_2^*\alpha_2 & 0 \\ z_3^*\alpha_1 & z_3^*\alpha_2 & 0 \end{bmatrix}.$$

We obtain $z_1^* = z_2^* = 0$ (since $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$) and $(e_{11} - e_{33})g_{13} = (e_{22} - e_{33})g_{23} = (e_{11} - e_{22})g_{12} = (e_{22} - e_{11})g_{21} = 0$. If $e_{11} = e_{22}$, then either $e_{11} = e_{22} = e_{33} = c$ so that $\mathbf{E} = c\mathbf{I}$ or $e_{11} \neq e_{33}$ and $e_{22} \neq e_{33}$. In the latter case, $g_{13} = g_{23} = 0$, and $\mathbf{G}_{31}$ has the first of the two forms of expression (8a); the model is reducible, contrary to assumption.

If $e_{11} \neq e_{22}$, we obtain $g_{12} = g_{21} = 0$. Moreover, we have either $e_{11} \neq e_{33}$ or $e_{22} \neq e_{33}$. In the first case $g_{13} = 0$, so that $g_{12} = 0$ and $g_{13} = 0$, and $\mathbf{G}_{31}$ takes on the second of the two forms given in expression (8a). In the second case, $g_{23} = 0$, and $\mathbf{G}_{31}$ takes on the first of the forms given in expression (8a). In both cases the model is reducible, contrary to assumption.

It remains to consider the possibility that $z_3 \neq 0$, in which case $e_{31} \neq 0$ and $e_{32} \neq 0$. Noting that $\mathbf{z}$ and $\boldsymbol{\alpha}$ are perpendicular, we see from Eq. (13b) that

$$[\mathbf{G}_{31}, \mathbf{E}]\mathbf{z} = \mathbf{z}^*\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{z} = \mathbf{0}.$$

It is easy to check that $\mathbf{E}\mathbf{z} = e_{33}\mathbf{z}$, and from the above equation it follows that $\mathbf{w} = \mathbf{G}_{31}\mathbf{z}$ is also an eigenvector of $\mathbf{E}$ corresponding to the eigenvalue e_{33}. If we use the fact that $e_{31} \neq 0$ and $e_{32} \neq 0$, there are just two possible forms for this eigenvector $\mathbf{w}$: $\mathbf{w} = k\mathbf{z}$, $k \neq 0$ or $\mathbf{w} = [a\, b\, c]^T$ with $a \neq 0, b \neq 0$, and $e_{31}a + e_{32}b = 0$. In the first case, $\mathbf{z}$ is an eigenvector of both $\mathbf{G}_{21}$ and $\mathbf{G}_{31}$, and it follows that the model is reducible ($g_{13} = g_{23} = 0$), contrary to assumption. The second possibility requires that $e_{11} = e_{22} = e_{33}$, and using this fact we obtain

$$[\mathbf{G}_{31}, \mathbf{E}] = \begin{bmatrix} e_{31}g_{13} & e_{32}g_{13} & 0 \\ e_{31}g_{23} & e_{32}g_{23} & 0 \\ e_{31}(g_{33} - g_{11}) - e_{32}g_{21} & e_{32}(g_{33} - g_{22}) - e_{31}g_{12} & e_{31}g_{13} + e_{32}g_{23} \end{bmatrix}$$
$$= \begin{bmatrix} z_1^*\alpha_1 & z_1^*\alpha_2 & 0 \\ z_2^*\alpha_1 & z_2^*\alpha_2 & 0 \\ z_3^*\alpha_1 & z_3^*\alpha_2 & 0 \end{bmatrix}.$$

We have, in particular, $z_1^*\alpha_1 = e_{31}g_{13}$, $z_1^*\alpha_2 = e_{32}g_{13}$, $z_2^*\alpha_1 = e_{31}g_{23}$, and $z_2^*\alpha_2 = e_{32}g_{23}$. If we use $z_3\alpha_1 = (\lambda_3 - \lambda_1)e_{31}$ and $z_3\alpha_2 = (\lambda_3 - \lambda_2)e_{32}$ [which follow from Eq. (15)], the first two of these four equations yield $z_3g_{13} = (\lambda_3 - \lambda_1)z_1^* = (\lambda_3 - \lambda_2)z_1^*$. Likewise, the second pair of equations yields $z_3g_{23} = (\lambda_3 - \lambda_1)z_2^* = (\lambda_3 - \lambda_2)z_2^*$. Since $\lambda_1 \neq \lambda_2$, it follows that $z_1^* = z_2^* = 0$ and hence $g_{13} = g_{23} = 0$: the model is reducible, contrary to assumption.

Case (c). $\alpha_1 \neq 0, \alpha_2 = 0, \alpha_3 = 0$. (The possibilities $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 \neq 0$, and $\alpha_1 = 0, \alpha_2 \neq 0, \alpha_3 = 0$ are notational variants of the case that we treat in detail.) We set $\alpha_1 = 1$ without loss of generality. From Eqs. (11) and (13a) we deduce that $e_{12} = e_{13} = e_{23} = e_{32} = 0$ and that $z_1 = 0$; $\mathbf{E}$ has the form

$$\mathbf{E} = \begin{bmatrix} e_{11} & 0 & 0 \\ e_{21} & e_{22} & 0 \\ e_{31} & 0 & e_{33} \end{bmatrix}.$$

The commutator $[\mathbf{G}_{21}, \mathbf{E}]$ is as follows:

$$[\mathbf{G}_{21}, \mathbf{E}] = \begin{bmatrix} 0 & 0 & 0 \\ z_2 & 0 & 0 \\ z_3 & 0 & 0 \end{bmatrix}. \tag{16}$$

We compute the commutator $[\mathbf{G}_{31}, \mathbf{E}]$ and obtain the following result:

$$[\mathbf{G}_{31}, \mathbf{E}] = \begin{bmatrix} e_{21}g_{12} + e_{31}g_{13} & (e_{22} - e_{11})g_{12} & (e_{33} - e_{11})g_{13} \\ x & -e_{21}g_{12} & -e_{21}g_{13} + (e_{33} - e_{22})g_{23} \\ x & -e_{31}g_{12} + (e_{22} - e_{33})g_{32} & -e_{31}g_{13} \end{bmatrix} \tag{17a}$$

$$= \begin{bmatrix} z_1^* & 0 & 0 \\ z_2^* & 0 & 0 \\ z_3^* & 0 & 0 \end{bmatrix}, \tag{17b}$$

where the entries marked x represent expressions that are not relevant for our present purpose. The second equality follows from Eq. (13b) and $\boldsymbol{\alpha} = [1\ 0\ 0]^T$.

If $e_{22} \neq e_{33}$ then $\mathbf{E}$ has either a unique eigenvector $[0\ 1\ 0]^T$, corresponding to the eigenvalue e_{22}, or a unique eigenvector $[0\ 0\ 1]^T$, corresponding to eigenvalue e_{33}. In the first case the eigenvector $[0\ 1\ 0]^T$ of $\mathbf{E}$ is also an eigenvector of $\mathbf{G}_{31}$, and it follows from Eq. (17a) that $g_{12} = g_{32} = 0$. In the second case the eigenvector $[0\ 0\ 1]^T$ of $\mathbf{E}$ is also an eigenvector of $\mathbf{G}_{31}$, and it follows that $g_{13} = g_{23} = 0$. In both cases the model is reducible, contrary to assumption.

The other possibility is that $e_{22} = e_{33}$. Comparing the second two columns of the matrices on the right-hand sides of Eqs. (17a) and (17b), we obtain (second column) $(e_{22} - e_{11})g_{12} = e_{21}g_{12} = e_{31}g_{12} = 0$ and (third column) $(e_{33} - e_{11})g_{13} = e_{21}g_{13} = e_{31}g_{13} = 0$. If $g_{12} \neq 0$ or $g_{13} \neq 0$, we obtain $\mathbf{E} = c\mathbf{I}$. Otherwise, $g_{12} = g_{13} = 0$ and the model is reducible, contrary to assumption.

(Sufficiency.) We assume that the model is reducible, so that $\mathbf{G}_{21}$ has the diagonal form given by Eq. (9), and $\mathbf{G}_{31}$ has one of the forms listed in expression (8a). Let us consider first the form on the right-hand side of expression (8a):

$$\mathbf{G}_{31} = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}.$$

Let us take $\mathbf{E}$ to be of the following diagonal form ($e_{22} = e_{33}; e_{11} \neq e_{33}$):

$$\mathbf{E} = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & e_{33} & 0 \\ 0 & 0 & e_{33} \end{bmatrix}.$$

Calculation shows that the two commutators $[\mathbf{G}_{21}, \mathbf{E}]$ and $[\mathbf{G}_{31}, \mathbf{E}]$ have a common form, namely,

$$\begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ x & 0 & 0 \end{bmatrix}.$$

This implies that Eq. (6) is satisfied with $\mathbf{E} \neq c\mathbf{I}$ when the illuminants lie in the subspace spanned by the vectors $[0\ 1\ 0]^T$ and $[0\ 0\ 1]^T$.

Finally, let us consider the case in which $\mathbf{G}_{31}$ has the form on the left-hand side of expression (8a), namely,

$$\mathbf{G}_{31} = \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{bmatrix}.$$

If $g_{31} = g_{32} = 0$, the model is decomposable, and we already know, from Theorem 1, that not all solutions $\mathbf{E}$ of Eq. (6) are multiples of identity matrix $\mathbf{I}$. Thus assume either that $g_{31} \neq 0$ or that $g_{32} \neq 0$. Take $\mathbf{E}$ to have the following diagonal form ($e_{22} = e_{11}; e_{11} \neq e_{33}$):

$$\mathbf{E} = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & e_{11} & 0 \\ 0 & 0 & e_{33} \end{bmatrix}.$$

Calculation shows that $[\mathbf{G}_{21}, \mathbf{E}] = \mathbf{0}$ and that

$$[\mathbf{G}_{31}, \mathbf{E}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -g_{31}(e_{33} - e_{11}) & -g_{32}(e_{33} - e_{11}) & 0 \end{bmatrix}.$$

From the latter expression we find that $[\mathbf{G}_{31}, \mathbf{E}]\mathbf{a}_1 = [\mathbf{G}_{31}, \mathbf{E}]\mathbf{a}_2 = \mathbf{0}$, where $\mathbf{a}_1 = [0\ 0\ 1]^T$ and $\mathbf{a}_2 = [-g_{32}\ g_{31}\ 0]^T$. The proof is complete. ■

DISCUSSION

We tested the criteria on the trichromatic bilinear models with components listed in Table 1.[1] For 3×3 model matrices, the case of interest, model reducibility is equivalent to the property that at least two associated model matrices (e.g., $\mathbf{G}_{21}$ and $\mathbf{G}_{31}$) share a common eigenvector, or at least two of the transposes of the associated model matrices (e.g., $\mathbf{G}_{21}{}^T$ and $\mathbf{G}_{31}{}^T$) share a common eigenvector. None of the 60 bilinear models, which we formed by choosing a set of spectral sensitivities, an illuminant basis, and a surface-reflectance basis, has this property.[2,3,13] We remind the reader that an irreducible model is necessarily indecomposable, so that all the models that we checked work perfectly to recover descriptors when either two or three views are provided. The positive results for these models agree with the results found with use of the model-check algorithm of D'Zmura and Iverson.[2,3]

The present criterion for three views provides both necessary and sufficient conditions for perfect recovery and

Table 1. Tested Bilinear Model Components

Photoreceptors
Smith and Pokorny[a]
Hurvich and Jameson[b]
CIE 10° observer[c]
Sony XC-007 CCD RGB
Sony XC-711 CCD RGB
Illuminants
Judd *et al.*[d]
Dixon[e]
Indoor (D_{65}, A, F_2[c]
Fourier
Reflectances
Cohen[f]
Parkkinen *et al.*[g]
Fourier

[a]Ref. 14.
[b]Ref. 15.
[c]Ref. 16.
[d]Ref. 8.
[e]Ref. 9.
[f]Ref. 6.
[g]Ref. 7.

so is equivalent, for regular models, to the model-check algorithm for three views.[2,3] For two views, the present criterion expresses necessity and sufficiency for regular models, representing an improvement over the model-check algorithm, which provides only sufficient conditions. The present criteria are more intuitive. They ensure that bilinear model matrices that must be different from one another do, in fact, differ appropriately. We formulate this intuition more precisely in terms of invariant properties of bilinear model matrices, namely, the properties of irreducibility and indecomposability. The numerical algorithm to check irreducibility is straightforward and, in our experience, suffices to classify trichromatic bilinear models for both two and three views.

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REFERENCES AND NOTES

1. M. D'Zmura, "Color constancy: surface color from changing illumination," J. Opt. Soc. Am. A **9,** 490–493 (1992).
2. M. D'Zmura and G. Iverson, "Color constancy. I. Basic theory of two-stage linear recovery of spectral descriptions for lights and surfaces," J. Opt. Soc. Am. A **10,** 2148–2165 (1993).
3. M. D'Zmura and G. Iverson, "Color constancy. II. Results for two-stage linear recovery of spectral descriptions for lights and surfaces," J. Opt. Soc. Am. A **10,** 2166–2180 (1993).
4. D. H. Marimont and B. A. Wandell, "Linear models of surface and illuminant spectra," J. Opt. Soc. Am. A **9,** 1905–1913 (1992).
5. G. J. Iverson and M. D'Zmura, "Criteria for color constancy in trichromatic bilinear models," presented at the Twenty-Fifth Annual Mathematical Psychology Meeting, Stanford University, Palo Alto, Calif., August 1992.
6. J. Cohen, "Dependency of the spectral reflectance curves of the Munsell color chips," Psychonom. Sci. **1,** 369–370 (1964).
7. J. P. S. Parkkinen, J. Hallikainen, and T. Jaaskelainen, "Characteristic spectra of Munsell colors," J. Opt. Soc. Am. A **6,** 318–322 (1989).
8. D. B. Judd, D. L. MacAdam, and G. Wyszecki, "Spectral distribution of typical daylight as a function of correlated color temperature," J. Opt. Soc. Am. **54,** 1031–1040 (1964).
9. E. R. Dixon, "Spectral distribution of Australian daylight," J. Opt. Soc. Am. **68,** 437–450 (1978).
10. L. T. Maloney and B. A. Wandell, "Color constancy: a method for recovering surface spectral reflectance," J. Opt. Soc. Am. A **3,** 29–33 (1986).
11. Note that our notion of decomposability is often referred to in the linear algebra literature as reducibility, e.g., P. R. Halmos, *Finite-Dimensional Vector Spaces*, 2nd ed. (Van Nostrand, Princeton, N.J., 1958), and S. K. Berberian, *Linear Algebra* (Oxford U. Press, New York, 1992). Our usage agrees with terminology used in the theory of group representations, e.g., M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Addison-Wesley, Reading, Mass., 1962).
12. S. Lang, *Linear Algebra*, 2nd ed. (Addison-Wesley, Reading, Mass., 1971).
13. E. Anderson, Z. Bai, C. Bischof, J. Demmel, J. Dongarra, A. DuCroz, S. Greenbaum, S. Hammarling, A. McKenney, S. Ostrouchov, and D. Sorensen, LAPACK User's Guide (SIAM, Philadelphia, 1992).
14. V. C. Smith and J. Pokorny, "Spectral sensitivity of the foveal cone photopigments between 400 and 500 nm," Vision Res. **15,** 161–171 (1975).
15. L. M. Hurvich and D. Jameson, "Some quantitative aspects of an opponent-colors theory. II. Brightness, saturation, and hue in normal and dichromatic vision," J. Opt. Soc. Am. **45,** 602–616 (1955).
16. G. Wyszecki and W. S. Stiles, *Color Science. Concepts and Methods, Quantitative Data and Formulas*, 2nd ed. (Wiley, New York, 1982).