Color constancy. III. General linear recovery of spectral descriptions for lights and surfaces

Michael D'Zmura and Geoffrey Iverson

Department of Cognitive Sciences and Institute for Mathematical Behavioral Sciences, University of California, Irvine, Irvine, California 92717

Received November 16, 1993; revised manuscript received April 21, 1994; accepted April 22, 1994

We present a color-constancy algorithm that uses quantum-catch data from reflected lights to recover surface reflectance functions and illuminant spectral power distributions. The algorithm recovers both surface and light-source spectral properties simultaneously. The method works in all situations that were handled by the earlier two-stage algorithms of Maloney and Wandell [J. Opt. Soc. Am. A **3**, 29 (1986)] and D'Zmura and Iverson [J. Opt. Soc. Am. A **9**, 490 (1992); **10**, 2148, 2166 (1993); **11**, 1970 (1994)]. In addition, the method handles problems that lie outside the scope of earlier algorithms. Using this method, a trichromatic visual system can recover, when provided adequate information, spectral descriptions of arbitrarily high accuracy for lights and surfaces. We determine conditions under which bilinear models can be used to recover color properties uniquely with the new procedure, and we formulate an algorithm for checking whether a particular bilinear model provides perfect color constancy. This research extends our analysis of linear methods for color constancy begun earlier [J. Opt. Soc. Am. A **10**, 2148, 2166 (1993)].

1. INTRODUCTION

Under simple viewing conditions the spectral properties of a reflected light depend on the product of the spectral power distribution of the light source and a surface's reflectance function. Variation in either light-source properties or surface composition induces variation in reflected light. A remarkable fact of human color vision is that a surface's color appearance is stable under conditions of varying illumination.[6–8] This phenomenon of color constancy has prompted the development of algorithms to estimate the spectral properties of a scene's illuminant and its surfaces.[9–20]

In two companion papers we analyzed how well two-stage linear recovery procedures can use bilinear models to determine surface and light-source chromatic properties from quantum-catch data.[3,4] The applicability of these two-stage procedures, like those of Maloney and Wandell[1] and D'Zmura,[2] is restricted by the need for the number of photoreceptoral types to equal or exceed either the dimension of the linear model for reflectance or the dimension of the linear model for illumination. A trichromatic visual system that uses a two-stage procedure is limited to recovering at most three spectral descriptors for either surfaces or light sources.

We present here a general linear recovery procedure that uses bilinear models to recover spectral descriptions of reflectances and illuminants simultaneously. The procedure works in situations in which the dimensions of the linear models for reflectance and illumination each may exceed the number of photoreceptoral types. A visual system that uses this new algorithm can recover, in principle, large numbers of spectral descriptors for both surfaces and light sources. The scope of the recovery procedure extends to all color constancy problems[3,4] in which both (1) the number of quantum-catch data compares favorably with the number of unknown spectral descriptors to be recovered and (2) the number of surfaces providing quantum-catch data compares favorably with the number of spectral descriptors per surface to be recovered.

It is important to identify which bilinear models can be used by a color constancy algorithm to recover the spectral descriptors of surfaces and light sources uniquely. The necessary and sufficient conditions for unique recovery remain the same as those for the two-stage procedures[3]: the bilinear model must provide a one-to-one relationship between quantum-catch data and sets of lit surfaces. We use this requirement to develop algorithms that check whether a given bilinear model, with the parameters of a particular color constancy problem, provides unique recovery.

We check the function of particular bilinear models in general linear recovery. These checks involve models for which the dimensions of both the reflectance and the illumination models exceed the number of photoreceptoral types. These tests of model function extend the classification of linear methods for color constancy initiated in the two companion papers.[3,4] We find that many views (provided by different light sources) of many surfaces are needed for recovering high-dimensional spectral descriptions of both surfaces and lights. We show that the general linear recovery algorithm lets p-chromatic systems, where $p \geq 2$, determine spectral descriptions of arbitrarily high dimension.

In this paper we follow the format of the preceding companion papers.[3,4] We first introduce the recovery algorithm (Section 2) and continue by discussing necessary conditions (criteria) for unique recovery to be possible (Section 3). The issue of whether a particular bilinear model does, in fact, provide unique recovery is not settled by these criteria. We are thus led, in Section 4, to provide a model check algorithm that possesses necessary and sufficient conditions. In Section 5 we check models and carry out further analysis in an effort to classify

0740-3232/94/092389-12$06.00

color constancy problems. This research was described elsewhere in preliminary form.[21]

2. GENERAL LINEAR RECOVERY PROCEDURE

We were led to develop the general linear recovery algorithm by noting that two-stage linear recovery procedures are restricted by the need to invert certain bilinear model matrices. In these two-stage procedures light-source descriptors are recovered in a first stage, and reflectance descriptors are recovered in a second stage,[1] or vice versa.[2] The matrix inverses are needed for carrying out these stages. In consequence, the dimensions of the recovered spectral descriptions for either lights or surfaces are restricted. However, such restrictions are not necessary. The new algorithm avoids the inversion of bilinear model matrices and so circumvents these restrictions. It recovers in a single stage both light-source and surface descriptors.

Each of the bilinear model matrices $\mathbf{B}_j$ for $j = 1, \ldots, n$ is of matrix dimension $p \times m$, in which p is the number of photoreceptoral types, m is the dimension of the linear model for illumination, and n is the dimension of the linear model for reflectance (see Table 1 and Refs. 2–5). Their entries $(\mathbf{B}_j)_{ki}$ are defined as follows [Eq. (8) of Ref. 1]:

$$(\mathbf{B}_j)_{ki} = \int Q_k(\lambda)A_i(\lambda)R_j(\lambda)\mathrm{d}\lambda\,, \qquad j = 1, \ldots, n, \quad (1)$$

in which appear the photoreceptoral spectral sensitivities $Q_k(\lambda)$, $k = 1, \ldots, p$; the illumination basis functions $A_i(\lambda)$, $i = 1, \ldots, m$; and the reflectance basis functions $R_j(\lambda)$, $j = 1, \ldots, n$. For a two-stage procedure to function, the bilinear model matrices must be of full rank, with $p \geq m$. The $p \times n$ model matrices $\mathbf{B}_i'$,[3] in which the roles of surfaces and illuminants are interchanged, provide a transposed recovery procedure that is constrained by the requirement that $p \geq n$. In a companion paper we show typical basis functions for illumination and reflectance as well as a set of photoreceptoral spectral sensitivities (Fig. 1 of Ref. 3).

Let us now describe the bilinear model for quantum-catch data in terms of model matrices and descriptors for illumination and reflectance. Consider the quantum catches received from a Mondrian[8] comprising s surfaces with different (i.e., for $s \leq n$, linearly independent) reflectance functions that is lit, in turn, by v illuminants. Each surface provides p quantum catches, so that the total number of quantum-catch data from v views of s surfaces is svp. Following D'Zmura and Iverson,[3] we introduce indices t and w, which run over the number s of surfaces and number v of views, respectively. Then the quantum catch q_{twk} of the kth photoreceptoral type produced by the tth surface viewed under the wth light is related to the n reflectance descriptors r_{tj}, for $j = 1, \ldots, n$, and the m illuminant descriptors a_{wi}, for $i = 1, \ldots, m$, in the following way [Eq. (10) of Ref. 3]:

$$q_{twk} = \sum_{j=1}^{n}\sum_{i=1}^{m} r_{tj}(\mathbf{B}_j)_{ki}a_{wi}\,. \quad (2)$$

This equation expresses the bilinear dependence of quantum-catch data on surface and illuminant descriptors.

Note that the system expressed by Eq. (2) is a bilinear, viz., nonlinear system. We will now work toward formulating an equivalent homogeneous linear system. We introduce the p-dimensional data vectors $\mathbf{d}_{tw} = [q_{tw1} \ldots q_{twp}]^T$ and m-dimensional vectors of illuminant descriptors $\mathbf{a}_w = [a_{w1} \ldots a_{wm}]^T$ [Eq. (11) of Ref. 3]:

$$\mathbf{d}_{tw} = \sum_{j=1}^{n} r_{tj}\mathbf{B}_j\mathbf{a}_w\,, \quad (3)$$

for $t = 1, \ldots, s$ and $w = 1, \ldots, v$. We define the pv-dimensional vectors $\mathbf{d}_t = [\mathbf{d}_{t1}{}^T \ldots \mathbf{d}_{tv}{}^T]^T$, the mv-dimensional vector $\mathbf{a} = [\mathbf{a}_1{}^T \ldots \mathbf{a}_v{}^T]^T$, and the $pv \times mv$ block-diagonal matrices

$$\mathbf{C}_j = \mathrm{diag}[\mathbf{B}_j, \ldots, \mathbf{B}_j]\,, \quad (4)$$

in which each of the blocks along the diagonal is $\mathbf{B}_j$; Eq. (3) thus takes on the form [Eq. (13) of Ref. 3]

$$\mathbf{d}_t = \sum_{j=1}^{n} r_{tj}\mathbf{C}_j\mathbf{a}, \qquad t = 1, \ldots, s\,. \quad (5)$$

Table 1. List of Symbols

Bilinear model parameters ($p\ m\ n\ v\ s$)	
p	Number of photoreceptor types
m	Illuminant model dimension
n	Reflectance model dimension
v	Number of views
s	Number of surfaces
Functions of wavelength	
$A(\lambda)$	An illuminant spectral power distribution
$A_i(\lambda)$	The ith illuminant model basis function, $i = 1, \ldots, m$
$L(\lambda)$	A reflected light
$Q_k(\lambda)$	The kth photoreceptor spectral sensitivity, $k = 1, \ldots, p$
$R(\lambda)$	A reflectance function
$R_j(\lambda)$	The jth reflectance model basis function, $j = 1, \ldots, n$
Descriptors, vectors, and matrices	
a_i, $\mathbf{a}_w$, $\mathbf{a}$, $\mathbf{A}$, $\mathbf{z}$, $\mathbf{Z}$	Illuminant descriptors
$\boldsymbol{\beta}_k$, $\mathbf{B}_j$	Bilinear model matrices
$\mathbf{C}_j$	Block-diagonal bilinear model matrices diag[$\mathbf{B}_j \ldots \mathbf{B}_j$]
$\boldsymbol{\delta}$	Vector of descriptors to be recovered
$\mathbf{d}_t$, $\mathbf{D}$, $\boldsymbol{\Delta}_k$	Quantum-catch data
e_{ij}, $\mathbf{e}$, $\mathbf{E}$	Variables relating two sets of reflectances
$\mathbf{F}$	Recovery matrix
$\mathbf{G}_{ij}$	Associated model matrices
$\mathbf{I}$	Identity matrix
$\mathbf{L}$	Model check matrix
ρ_{jt}, ρ_j, ρ, $\mathbf{P}$	Inverse of the matrix of reflectance descriptors
r_j, $\mathbf{r}_t$, r_{tj}, $\mathbf{R}$, σ_{it}, $\mathbf{S}$	Reflectance descriptors
Miscellany	
e, u	Numbers of equations and monomial unknowns, respectively, for the generalized invertibility criterion
E, U	Numbers of equations and monomial unknowns, respectively, provided by a model check algorithm
Q, D	Number svp of quantum-catch data and number $sn + vm$ of unknown descriptors to be recovered

When $s = n$, the descriptors r_{tj} can be regarded as the elements of a square $n \times n$ matrix $\mathbf{R}$. If the n surfaces viewed are different, i.e., the vectors formed by each surface's reflectance descriptors are linearly independent, then the reflectance matrix $\mathbf{R}$ is nonsingular, with inverse

$$\mathbf{P} = \mathbf{R}^{-1}. \tag{6}$$

When ρ_{jt} is used to label the entries of the matrix $\mathbf{P}$, it follows that Eq. (5) can be rewritten in the form

$$\sum_{t=1}^{n} \rho_{jt}\mathbf{d}_t - \mathbf{C}_j\mathbf{a} = \mathbf{0}, \qquad j = 1, \ldots, n. \tag{7}$$

We have here the desired system of linear homogeneous equations. Note that the bilinear model matrices, represented by the matrices $\mathbf{C}_j$, $j = 1, \ldots, n$, have not been inverted. In consequence, the number p of photoreceptors need not match or exceed the dimension m of the illumination model, in contrast to the case with the earlier two-stage algorithms.[1–5]

It remains to write the system of Eq. (7) in a more convenient form, one in which the unknown descriptors can be recovered by determination of the one-dimensional kernel of a recovery matrix.[2–4] Let us introduce the $n \times 1$ vectors

$$\boldsymbol{\rho}_j = [\rho_{j1} \ldots \rho_{jn}]^T, \qquad j = 1, \ldots, n, \tag{8}$$

which may be stacked, in order, to form a single $n^2 \times 1$ vector

$$\boldsymbol{\rho} = [\boldsymbol{\rho}_1{}^T \ldots \boldsymbol{\rho}_n{}^T]^T. \tag{9}$$

The vector $\boldsymbol{\rho}$ and the vm-dimensional vector $\mathbf{a} = [a_{11} \ldots a_{vm}]^T$ of illuminant descriptors form a complete list of all unknown descriptors. It is convenient to form a single $(n^2 + vm)$-dimensional vector $\boldsymbol{\delta}$ from these:

$$\boldsymbol{\delta} = \left[\frac{\boldsymbol{\rho}}{\mathbf{a}}\right]. \tag{10}$$

We use the pv-dimensional data vectors $\mathbf{d}_t$, $t = 1, \ldots, n$, introduced above in Eq. (5), to form the following $pv \times n$ matrix $\mathbf{D}$, which records all pnv quantum catches:

$$\mathbf{D} = [\mathbf{d}_1 \ldots \mathbf{d}_n]. \tag{11}$$

From the matrices $\mathbf{D}$ and $\mathbf{C}_j$, $j = 1, \ldots, n$, can be formed the partitioned $pnv \times (n^2 + vm)$ recovery matrix $\mathbf{F}$:

$$\mathbf{F} = \left[\begin{array}{cccc|c} \mathbf{D} & \mathbf{0} & \ldots & \mathbf{0} & -\mathbf{C}_1 \\ \mathbf{0} & \mathbf{D} & & \mathbf{0} & -\mathbf{C}_2 \\ \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & & \mathbf{D} & -\mathbf{C}_n \end{array}\right], \tag{12}$$

in which all entries are zero other than the n blocks of data along the diagonal and the n blocks of bilinear model matrices for multiple views in the last block column. The homogeneous system of linear equations defined in Eq. (7) can now be written compactly as

$$\mathbf{F}\boldsymbol{\delta} = \mathbf{0}. \tag{13}$$

This is the final form of the linear homogeneous system. The unknown descriptors are the components of the vector $\boldsymbol{\delta}$, and the quantum-catch data and the bilinear model matrix entries determine the recovery matrix $\mathbf{F}$.

The matrix $\mathbf{F}$ is the key to recovery. It depends in part on data and in part on the numerical structure of a bilinear model, represented by the matrices $\mathbf{C}_j$, $j = 1, \ldots, n$. From elementary linear algebra, we see at once that unique recovery, up to an arbitrary positive scalar, is possible if and only if $\mathbf{F}$ possesses a one-dimensional null space, i.e.,

$$\dim[\ker(\mathbf{F})] = 1. \tag{14}$$

A necessary condition for this to be true is that the number of rows in the matrix $\mathbf{F}$ equal or exceed the number of its columns, minus one, a requirement that returns the feasibility condition of D'Zmura and Iverson [inequality (14) of Ref. 3] under the single restriction $s = n$:

$$pnv \geq n^2 + mv - 1. \tag{15}$$

The general linear recovery algorithm is thus as follows: (1) use quantum-catch data and the bilinear model to determine the matrix $\mathbf{F}$ [Eq. (12)]; (2) find the kernel of the matrix, where reside the illuminant and (inverse) reflectance descriptors.

The kernel of the recovery matrix $\mathbf{F}$ can be determined numerically with a singular value decomposition, which provides a basis for the null space of a singular matrix.[22] In the event that (1) the matrix $\mathbf{R}$ of reflectance descriptors is invertible, (2) the v vectors of illuminant descriptors are linearly independent, (3) the feasibility condition [inequality (15)] is satisfied, and (4) conditions on the matrices $\mathbf{B}_j$ developed in Sections 3 and 4 below are met, than the singular value decomposition of $\mathbf{F}$ returns a single nonzero vector $\boldsymbol{\delta}$ that contains the descriptors ρ_{jt} and a_{wi} up to an arbitrary common scale; scaled reflectance descriptors r_{tj} are recovered from the ρ_{jt} by matrix inversion. Note that the algorithm recovers the descriptors up to an arbitrary scalar, as it cannot discriminate reflectances $\mathbf{R}$ lit by illuminants $\mathbf{A}$ from the scaled reflectances $\kappa\mathbf{R}$ lit by the reciprocally scaled illuminants $(1/\kappa)\mathbf{A}$.

3. CRITERIA FOR GENERAL LINEAR RECOVERY

There are two natural classes of color constancy problems to consider. One involves two views of a set of surfaces. Such a problem is $(p\ m\ n\ v\ s) = (3\ 3\ 3\ 2\ 3)$, in which a trichromatic visual system attempts to recover three descriptors for each illuminant and surface when provided data from two views of the three surfaces.[2–5] The cases that involve two views arise in situations in which the illumination of a set of surfaces changes. This change can occur in time, so that two different illuminants shine in succession on a set of surfaces. This type of change was used by Land[8] in his demonstrations of human color constancy. The change in illumination can also occur across space, as in the common situation in which outdoor surfaces, in partial shadow, are lit simultaneously by bluish skylight and by the yellowish light from the solar disk. The other class of natural color constancy problems involves a single view of a set of surfaces.[1]

The general linear recovery procedure applies to almost every feasible problem in these two natural classes. Indeed, general linear recovery applies to all problems handled by the earlier two-stage linear recovery algorithms.[1–5] The feasibility criterion for general linear recovery [inequality (15)], which was developed under the restriction that the number s of surfaces equal (or exceed) the dimension n of the reflectance model, subsumes the problems feasible for two-stage linear recovery, as these require the further restriction that the number of photoreceptors p equal or exceed the dimension m of the model for illumination and/or the dimension n of the model for reflectance.

However, our primary focus in this paper will be on problems in which two-stage recovery is impossible, namely, on those in which the number p of photoreceptors is less than the dimensions m and n for illumination and reflectance. In what follows we first examine necessary conditions (criteria) for unique recovery to be possible. In Subsection 3.A we discuss briefly the restriction $s \geq n$. In Subsection 3.B we derive a criterion that rules out problems in which there exist illuminants that are invisible to bilinear models with the problem's parameters. We extend this criterion in Subsection 3.C to a more general test of whether there exist sets of lit surfaces that are invisible. This test of a particular bilinear model generalizes the necessary condition, met in our analysis of two-stage linear recovery,[3] that bilinear model matrices be of full rank. In Subsection 3.D we derive a further inequality among problem parameters that extends the criterion $pv > n$ for two-stage recovery to problems where $p < m, n$. Finally, we consider the transposition of general linear recovery and resulting criteria in Subsection 3.E.

A. Restriction $s \geq n$

The first necessary condition is that the number s of surfaces equal or exceed the dimension n of the model for reflectance. To see this, note that for linear recovery to be possible the reflectance descriptors r_{tj} in Eq. (5) must form an $s \times n$ matrix $\mathbf{R}$ of full rank: a unique left inverse for $\mathbf{R}$ with entries ρ_{jt} is needed for unique recovery. No linear scheme can recover reflectance descriptors if $s < n$, because the absence of unique inverses for (1) the $s \times n$ matrix $\mathbf{R}$ of reflectance descriptors and (2) the $p \times m$ model matrices $\mathbf{B}_j$, $j = 1, \ldots, n$, disallows manipulation that would provide a system of linear equations. The transposed criterion for illuminant descriptors is discussed in Subsection 3.E below.

B. Criterion $p + v > m$

A further necessary condition for unique recovery is expressed by the inequality

$$p + v > m. \tag{16}$$

In words, the sum of the number p of photoreceptoral types and the number v of views must exceed the dimension m of the model for illumination. If, for some particular problem, this inequality does not hold true, then one can find invisible illuminants that cause recovery procedures with the problem's parameters to fail.

Note that in problems where $p \geq m$, inequality (16) holds automatically. We argue here that the criterion is a necessary condition in problems where $p < m$. The block-diagonal matrices $\mathbf{C}_j$ of Eqs. (4) and (5) are of matrix dimension $pv \times mv$, and if $p < m$, then $\dim[\ker(\mathbf{C}_j)] > v(m - p)$ for $j = 1, \ldots, n$. If there exists a vector of illuminant descriptors $\mathbf{a} = [\mathbf{a}_1{}^T \ldots \mathbf{a}_v{}^T]^T$ with linearly independent constituents $\mathbf{a}_1, \ldots, \mathbf{a}_v$ such that

$$\mathbf{a} \in \bigcap_{j=1}^{n} \ker(\mathbf{C}_j), \tag{17}$$

then recovery must fail: Eq. (5) shows that the quantum-catch data are identically zero for such illuminants.

The condition on the bilinear model matrix constituents $\mathbf{B}_j$ of the block-diagonal matrices $\mathbf{C}_j$ for such invisible illuminants to exist is that $\mathbf{B}_j\mathbf{a}_w = \mathbf{0}$ for $w = 1, \ldots, v$. This undesirable situation can occur if $\dim[\ker(\mathbf{B}_j)] \geq v$, and we must rule out this possibility. By hypothesis, $\dim[\ker(\mathbf{B}_j)] \geq m - p$, so we insist that $m - p < v$, which, on rearrangement, produces the criterion inequality (16).

C. Generalized Invertibility Criterion on Model Matrices When $p < m$

Although inequality (16) lets us eliminate color-constancy problems for which recovery must fail, we can derive stronger conditions by ruling out the possibility that there are invisible surfaces. If in Eq. (5) there is some illuminant vector $\mathbf{a}^0$ for which the vectors $\mathbf{C}_1\mathbf{a}^0, \ldots, \mathbf{C}_n\mathbf{a}^0$ are not linearly independent, then there are surfaces that give rise to zero quantum catches under $\mathbf{a}^0$. Such invisible surfaces can be superimposed upon any solution r_{tj} of Eq. (5). That is, there exist reflectance descriptors $r_j{}^0$, $j = 1, \ldots, n$, not all zero, such that

$$\sum_{j=1}^{n} r_j{}^0 \mathbf{C}_j \mathbf{a}^0 = \mathbf{0}. \tag{18}$$

In matrix form this equation reads as

$$\sum_{j=1}^{n} r_j{}^0 \mathbf{B}_j \mathbf{A}^0 = \mathbf{0}, \tag{19}$$

where the v columns of $\mathbf{A}^0$ hold the vectors of illuminant descriptors $\mathbf{a}_w{}^0$, $w = 1, \ldots, v$, which are linearly independent, by hypothesis. We thus infer from Eq. (19) that the p rows of the $p \times m$ matrix $\sum_{j=1}^{n} r_j{}^0 \mathbf{B}_j$ lie in the subspace of dimension $m - v$ that is orthogonal to the v columns of $\mathbf{A}^0$. In analogy to the derivation of the model-check algorithm for two-stage linear recovery,[3] we see that all subdeterminants of $\sum_{j=1}^{n} r_j{}^0 \mathbf{B}_j$ of size $m - v + 1$ must vanish. This provides $\mathfrak{e}$ homogeneous polynomial equations of degree $m - v + 1$ in the n unknowns $r_j{}^0$, $j = 1, \ldots, n$, where $\mathfrak{e}$ is the product of two binomial coefficients:

$$\mathfrak{e} = \binom{p}{m - v + 1}\binom{m}{m - v + 1}. \tag{20}$$

To rule out the possibility of invisible surfaces, it is necessary and sufficient that these equations have only the trivial, zero solution:

$$r_j^{\,0} = 0, \qquad j = 1, \ldots, n. \tag{21}$$

Note that the number of equations must be positive, and examining the first binomial coefficient shows us that $p \geq m - v + 1$, which is equivalent to inequality (16). When $m > v$, we can linearize the equations[3] by expressing them in terms of the distinct monomials of degree $m - v + 1$, and the number u of these unknowns is

$$\mathsf{u} = \binom{n + m - v}{m - v + 1}. \tag{22}$$

Thus, to rule out invisible surfaces, it is sufficient that this linearized system of homogeneous equations have no nontrivial solution, in which case the only invisible surface is, in fact, the null, zero surface. This criterion generalizes the requirement, posed earlier for two-stage linear recovery,[3,4] that the bilinear model matrices be of full rank.

D. Criterion $pv > n + 1$ for Problems Where $p < m$

We showed[3] that the inequality $pv > n$ is a necessary condition for two-stage linear recovery. This inequality states that the product of the number p of photoreceptors and the number v of views must exceed the dimension n of the model for reflectance. It generalizes to multiple views the inequality $p > n$ for single views proposed by Maloney and Wandell.[1] Yet the inequality $pv > n$ depends on the assumption that $p \geq m$, and it happens that we can strengthen the inequality in problems where $p < m$.

Substituting $p < m$ into inequality (15), we find that

$$pnv > pv + n^2 - 1, \tag{23}$$

and, subtracting pv from both sides, we find that

$$pv(n - 1) > (n - 1)(n + 1). \tag{24}$$

In all cases of interest, the quantity $n - 1$ is a positive integer, and inequality (24) simplifies to

$$pv > n + 1. \tag{25}$$

This is a necessary condition on color constancy problem parameters when the number p of photoreceptors is less than the dimension m of the model for illumination.

E. Transposition

As is the case for two-stage linear recovery,[3] the roles of surfaces and light sources may be interchanged in the general recovery procedure. This interchange leads to a restatement of the inequalities on the problem parameters presented above. In particular, one needs a sufficient number of independent views: $v \geq m$. Furthermore, the criterion $p + v > m$ becomes $p + s > n$ under transposition, and the criterion $pv > n + 1$ when $p < m$ becomes $ps > m + 1$ when $p < n$.

4. MODEL CHECK ALGORITHM

A recovery algorithm with parameters that meet the feasibility criteria presented in Section 3 need not work. As we argued in the companion papers,[3,4] such criteria involve restrictions mainly on the parameters $(p\ m\ n\ v\ s)$ of a color-constancy problem. Other than ensuring invertibility, the criteria ignore the structure of the bilinear model matrices [Eq. (1)]. Furthermore, the criteria do not take into account the structure of recovery matrices $\mathbf{F}$ [Eq. (12)], which are crucial to the recovery algorithm.

For recovery to be possible for all valid sets of quantum-catch data, it is necessary and sufficient for a bilinear model to provide a one-to-one relationship between sets of lit surfaces and quantum-catch data, up to an arbitrary positive scalar.[3] In cases in which the number p of photoreceptoral types equals or exceeds the dimension m of the model for illumination, formalizing the requirement for a one-to-one relationship leads to a model check algorithm[3] for two-stage linear recovery. The model check algorithm is used to test whether a particular bilinear model provides a recovery algorithm that works perfectly with appropriate data.[4]

The need for a one-to-one relationship between quantum-catch data and sets of lit surfaces also leads to model check algorithms for general linear recovery. The algorithms work in situations in which the number p of photoreceptors is less than both the dimension m of the model for illumination and the dimension n of the model for reflectance. In Subsection 4.A we present a linear model check algorithm that expresses necessary and sufficient conditions for perfect recovery in cases in which the number v of views equals (or exceeds) the dimension m for illumination. In Subsection 4.B we discuss the nonlinear problem met in situations in which the number of views is fewer ($v < m$).

A. Necessary and Sufficient Check for Problems Where $p < m$ and $v = m$

We use the methods introduced in the companion papers[3,4] to formulate necessary and sufficient conditions for unique recovery by bilinear models in cases in which the number v of views matches the dimension m of the model for illumination. The result is a model check algorithm that works by (1) using a bilinear model to create a model check matrix and (2) checking the dimension of the kernel of the model check matrix. If the dimension is one, then the bilinear model provides unique recovery.

Let us first rewrite Eq. (3) as

$$\mathbf{D}_t = \sum_{j=1}^{n} r_{tj}\mathbf{B}_j\mathbf{A}, \qquad t = 1, \ldots, n, \tag{26}$$

by taking the illuminant descriptors and quantum-catch data from each view to fill a column of the matrices $\mathbf{A}$ and $\mathbf{D}$, respectively. The matrix $\mathbf{A}$ of illuminant descriptors has dimension $m \times v$, the bilinear model matrices are $p \times m$, and the data matrices are $p \times v$. Now suppose that a second set of surfaces lit by a second set of lights gives identical data. By Eq. (26),

$$\mathbf{D}_t = \sum_{i=1}^{n} s_{ti}\mathbf{B}_i\mathbf{Z} = \sum_{j=1}^{n} r_{tj}\mathbf{B}_j\mathbf{A}, \qquad t = 1, \ldots, s = n. \tag{27}$$

If recovery is to be unique, then the only way for the two sets of surfaces under their respective light sources to give rise to identical data is for the reflectance descriptors r_{tj}

and s_{tj} to be identical, up to some scale factor, and for the illuminant descriptor matrices $\mathbf{A}$ and $\mathbf{Z}$ to be identical, up to the reciprocal scale factor.

By hypothesis ($s = n$), the matrix $\mathbf{S}$ of reflectance descriptors s_{tj} is invertible, so leading from Eq. (27) to

$$\mathbf{B}_i\mathbf{Z} = \sum_{j=1}^{n} e_{ij}\mathbf{B}_j\mathbf{A}, \qquad i = 1, \ldots, n, \tag{28}$$

where

$$e_{ij} = \sum_{t=1}^{n} \sigma_{it} r_{tj}, \tag{29}$$

in terms of the elements σ_{it} of the matrix $\mathbf{S}^{-1}$ [Eq. (36) of Ref. 3]. Multiplying Eq. (28) on the right-hand side by $\mathbf{A}^{-1}$, we find that

$$\mathbf{B}_i\mathbf{H} = \sum_{j=1}^{n} e_{ij}\mathbf{B}_j, \qquad i = 1, \ldots, n, \tag{30}$$

where

$$\mathbf{H} = \mathbf{Z}\mathbf{A}^{-1}. \tag{31}$$

Subtracting from each of the n Eqs. (30) its left-hand-side leads to the homogeneous system of equations

$$\begin{gathered}\sum_{j=1}^{n} e_{1j}\mathbf{B}_j - \mathbf{B}_1\mathbf{H} = \mathbf{0},\\ \vdots \\ \sum_{j=1}^{n} e_{nj}\mathbf{B}_j - \mathbf{B}_n\mathbf{H} = \mathbf{0}.\end{gathered} \tag{32}$$

This system provides n blocks of pm linear equations in the $n^2 + m^2$ variables e_{ij} and h_{ij}, where the latter m^2 variables h_{ij} are the elements of the matrix $\mathbf{H}$ of Eq. (31).

The system can be written in the compact form

$$\mathbf{L}\boldsymbol{\omega} = \mathbf{0}, \tag{33}$$

where the matrix $\mathbf{L}$ has dimensions $npm \times (n^2 + m^2)$ and the vector $\boldsymbol{\omega}$ has dimension $n^2 + m^2$. The vector $\boldsymbol{\omega}$ is defined to be

$$\boldsymbol{\omega} = [e_{11} \ldots e_{1n} \ldots e_{n1} \ldots e_{nn} \quad h_{11} \ldots h_{1m} \ldots h_{m1} \ldots h_{mm}]^T. \tag{34}$$

Continue by defining the matrix $\mathbf{X}$, of dimension $pm \times n$, and the matrices $\mathbf{Y}_i$, $i = 1, \ldots, n$, each of dimension $pm \times m^2$:

$$\mathbf{X} = \begin{bmatrix} (\mathbf{B}_1)_{11} & \cdots & (\mathbf{B}_n)_{11} \\ & \vdots & \\ (\mathbf{B}_1)_{1m} & \cdots & (\mathbf{B}_n)_{1m} \\ \hdashline & \vdots & \\ \hdashline (\mathbf{B}_1)_{p1} & \cdots & (\mathbf{B}_n)_{p1} \\ & \vdots & \\ (\mathbf{B}_1)_{pm} & \cdots & (\mathbf{B}_n)_{pm} \end{bmatrix}, \tag{35}$$

$$\mathbf{Y}_i = \begin{bmatrix} [-(\mathbf{B}_i)_{11} \cdots -(\mathbf{B}_i)_{1m}] & & \mathbf{0} \\ & \ddots_m & \\ \mathbf{0} & & [-(\mathbf{B}_i)_{11} \cdots -(\mathbf{B}_i)_{1m}] \\ \hdashline & \vdots & \\ \hdashline [-(\mathbf{B}_i)_{p1} \cdots -(\mathbf{B}_i)_{pm}] & & \mathbf{0} \\ & \ddots_m & \\ \mathbf{0} & & [-(\mathbf{B}_i)_{p1} \cdots -(\mathbf{B}_i)_{pm}] \end{bmatrix}. \tag{36}$$

Use the matrices $\mathbf{X}$ and $\mathbf{Y}_i$, $i = 1, \ldots, n$, to construct the model check matrix $\mathbf{L}$ of Eq. (33) as follows:

$$\mathbf{L} = \left[\begin{array}{ccc|c} \mathbf{X} & & \mathbf{0} & \mathbf{Y}_1 \\ & \ddots_n & & \vdots \\ \mathbf{0} & & \mathbf{X} & \mathbf{Y}_n \end{array}\right]. \tag{37}$$

From Eq. (33) we see that the necessary and sufficient condition for a bilinear model to recover descriptors uniquely, in cases where $p < m$ and $m = v$, is that

$$\dim[\ker(\mathbf{L})] = 1. \tag{38}$$

In the case in which the kernel of model check matrix $\mathbf{L}$ is of dimension one, then the vector $\boldsymbol{\omega}$ of Eq. (34) contains only the scaling solutions:

$$\begin{aligned} e_{11} - e_{jj} &= 0 \qquad \text{for } j = 2, \ldots, n, \\ e_{ij} &= 0 \qquad \text{for } i \neq j, \\ h_{11} - h_{jj} &= 0 \qquad \text{for } j = 2, \ldots, m, \\ h_{ij} &= 0 \qquad \text{for } i \neq j, \\ e_{11} &= h_{11}. \end{aligned} \tag{39}$$

The linear model check algorithm, then, tests whether the $npm \times (n^2 + m^2)$ matrix $\mathbf{L}$ of Eq. (37) has a one-dimensional kernel. The algorithm is based on a singular value decomposition[4] of the model check matrix $\mathbf{L}$. Passing this test is both necessary and sufficient for a particular bilinear model to recover descriptors perfectly. In Section 5 below we present results of applying this algorithm to particular bilinear models.

B. Sufficient Checks for Problems Where $p < m$ and $v < m$

In cases in which the number v of views is less than the dimension m of the illumination model, the necessary and sufficient conditions for unique recovery lead to a set of nonlinear equations. This holds true in the situation of present concern, namely, general linear recovery in cases where $p < m$, and also for situations in which two-stage linear recovery procedures work.[3,4] However, we find that we can linearize the nonlinear equations to produce model check algorithms that express sufficient (but not necessary) conditions for unique recovery.

We have found two such algorithms for the class of problems in which $p < m$. The first follows from developments like those in the companion papers.[3,4] Unfortunately, the homogeneous systems of linear equations that are produced by this model check algorithm are very large. This fact prompted us to apply the analysis of Iverson and D'Zmura[5] to the present problem. In cases in which the dimension n of the model for reflectance equals or exceeds the dimension m of the model for illumination, a second model check algorithm results that produces more compact systems of equations. The second algorithm also has limited applicability, as discussed below. We make available both of these algorithms through a technical report.[23]

5. RESULTS

Here we present results of classifying color constancy problems that are met when (1) the number p of photoreceptors is less than both the dimensions m and n of the models for illumination and reflectance and (2) the number s of surfaces equals or exceeds the dimension n of the reflectance model. The difficulty in performing model checks in cases in which the number v of views is less than the dimension m of the illumination model leads us to limit our scope to those problems where $v = m$, for which we have a necessary and sufficient model check algorithm (Subsection 4.A). For these problems we exhibit bilinear models with the problem parameters that can be used to recover spectral descriptions uniquely, when possible. We follow Section 2 (Methods) of the companion paper,[4] to which the reader is referred.

We consider, in turn, dichromatic problems (Subsection 5.A), trichromatic problems (Subsections 5.B and 5.C), and, more generally, p-chromatic problems, for $p \geq 2$ (Subsection 5.D). We show the results of checks of particular bilinear models (Subsection 2.B of Ref. 4) with components listed in Table 2, together with the spectra of singular values of exemplary model-check matrices (Subsection 2.C of Ref. 4).

A. Dichromacy

Figure 1 shows results for general linear recovery of spectral descriptions by dichromatic systems. The format of the diagrams follows that of similar figures in the companion paper (e.g., Fig. 1 of Ref. 4) and is detailed in the caption.

Results for problems with a three-dimensional illumination model are shown in Fig. 1A. The positive result for the problem with parameters (p m n v s) = (2 3 2 3 2), indicated by the circle, follows from transposition [entailment (d) of Ref. 4) of the positive result[4] for (2 2 3 2 3). The problem with parameters (2 3 3 3 3) fails totally, as indicated by the **X**. This total failure is suggested by the failure of the model-check algorithm for the 36 models with these parameters that can be formed from the appropriate components of Table 2. The total failure of (2 3 3 3 3) can be shown analytically: the proof[3] of the failure of models of the form (2 2 2 2 2) is readily extended to show that all models of form (2 c c c c), $c \geq 2$, also fail totally. We leave this proof to the reader. Figure 1A indicates, finally, that there are

Table 2. Tested Bilinear Model Components

Photoreceptors	Illuminants	Reflectances
Two dimensions		
Smith–Pokorny[24] protanope	Judd *et al.*[25]	Cohen[26]
Smith–Pokorny deuteranope	Dixon[27]	Parkkinen *et al.*[28]
Smith–Pokorny tritanope	Fourier (1, sin)	Fourier (1, sin)
	Fourier (1, cos)	
Three dimensions		
Smith and Pokorny[24]	Judd *et al.*[25]	Cohen[26]
Hurvich and Jameson[29]	Dixon[27]	Parkkinen *et al.*[28]
CIE 10° observer[30]	Indoor (D_{65}, A, F_2)[30]	Fourier
Stockman *et al.* 2°–10° observer[31]	Fourier	
Sony XC-007 CCD RGB		
Sony XC-711 CCD RGB		
Four dimensions		
	Judd *et al.*[25]	Cohen[26]
	Dixon[27]	Parkkinen *et al.*[28]
	Fourier	Fourier
Five dimensions		
	Fourier	Parkkinen *et al.*[28]
		Fourier
Six+ dimensions		
	Fourier	Fourier

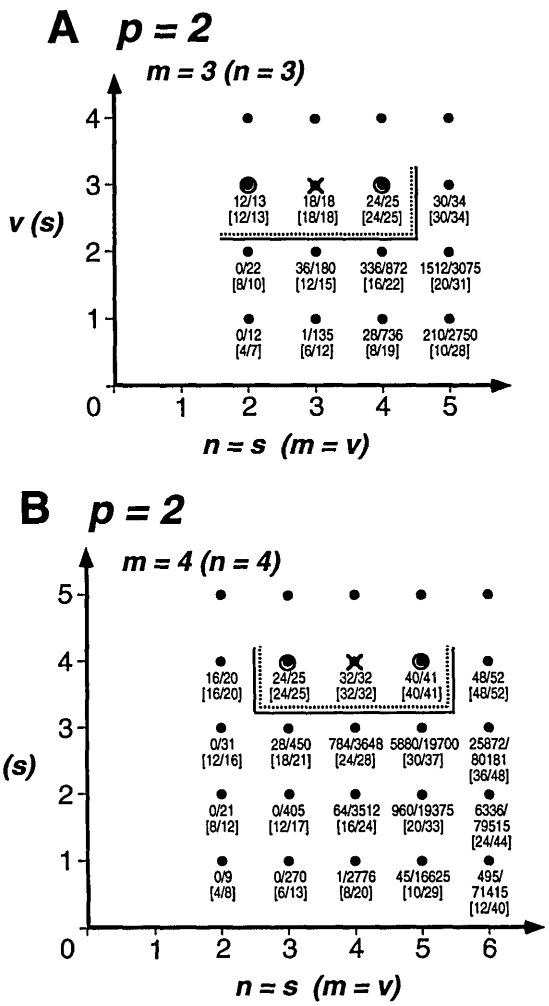


Fig. 1. Results for dichromatic bilinear models: A, the case in which the illumination model has dimension three ($p = 2$, $m = 3$ or by transposition $p = 2$, $n = 3$); B, the case in which the illumination model has dimension four ($p = 2$, $m = 4$ or by transposition $p = 2$, $n = 4$). The format is like that used in Fig. 1 of Ref. 4. The horizontal axes mark the dimension n of the reflectance model, which is taken equal to the number s of surfaces, and the vertical axes mark the number v of views. The solid lines divide cases that satisfy the necessary conditions $pnv \geq n^2 + vm - 1$ [inequality (15)] and $p + v > m$ [inequality (16)]. The number of quantum-catch data $\mathsf{Q} = svp$ and the number $\mathsf{D} = sn + vm$ of spectral descriptors to be recovered are indicated for each problem by the bracketed pair [Q/D] beneath the appropriate point. The dotted lines divide cases that satisfy the necessary condition for the sufficient test provided by the first nonlinear model-check algorithm, namely, that $\mathsf{E}_1 \geq \mathsf{U}_1$ [Eqs. (A5) and (A11) of Ref. 23]. The pair $\mathsf{E}_1/\mathsf{U}_1$ is shown directly beneath each point. In cases where $m = v$ (top rows), a necessary and sufficient model check examines the rank of a system of npm equations in $n^2 + m^2$ unknowns [Eq. (33)]; these two values are shown directly beneath such points. An X marks problems for which recovery fails totally. The circled points mark problems for which there are perfect recovery algorithms. Parameters for transposed problems are indicated in parentheses at the tops of the diagrams and along their axes.

bilinear models with parameters (2 3 4 3 4) that function perfectly.

In Fig. 1B are shown results for dichromatic problems with a four-dimensional model of illumination. The positive result for the problem (2 4 3 4 3) follows from transposition of the positive result shown in Fig. 1A for the problem (2 3 4 3 4). The total failure for (2 4 4 4 4) is again suggested by the failure of model checks and is an instance of the failure of (2 c c c c), $c \geq 2$. Finally, the model check algorithm shows that there are bilinear models with parameters (2 4 5 4 5) that work perfectly.

In Fig. 2 are shown spectra of ordered singular values of model check matrices for exemplary bilinear models. As described in Section 2 of Ref. 4, one determines such a spectrum by computing a singular value decomposition of a model check matrix and ordering the singular values from greatest to least. We then check whether there is a plunge between adjacent values in the spectrum of ordered singular values of five log units or greater. We take such a plunge to indicate the presence of a nontrivial kernel, and the number of small singular values indicates the rank of the kernel.

Figure 2 shows the spectra for exemplary models with parameters (2 3 3 3 3), (2 3 4 3 4), (2 4 4 4 4), and (2 4 5 4 5), from bottom to top. The kernels of the matrices are dimensions three, one, four, and one, respectively. A model passes the necessary and sufficient test posed by the model check algorithm (Subsection 4.A) if its matrix has a kernel of dimension one, so that these spectra indicate failure for (2 3 3 3 3) and (2 4 4 4 4) and success for (2 3 4 3 4) and (2 4 5 4 5). Although all problems of form (2 c c c c) must fail, the numerical results suggest that there may be perfect recovery procedures for all problems of the form (2 c $c+1$ c $c+1$) or, by transposition, of the form (2 $c+1$ c $c+1$ c). We investigate this possibility more fully in Subsection 5.D below.

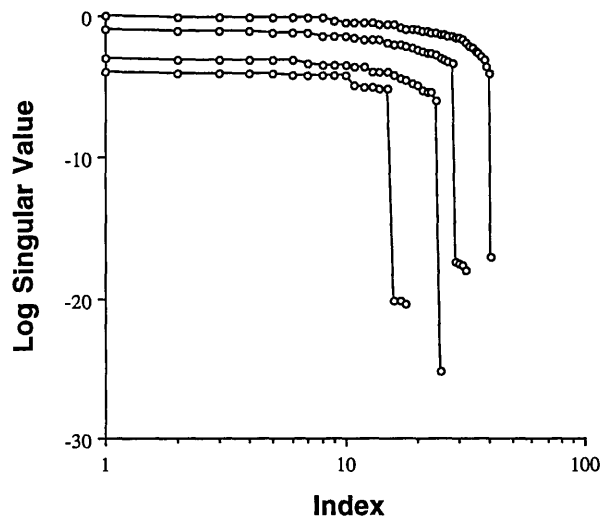
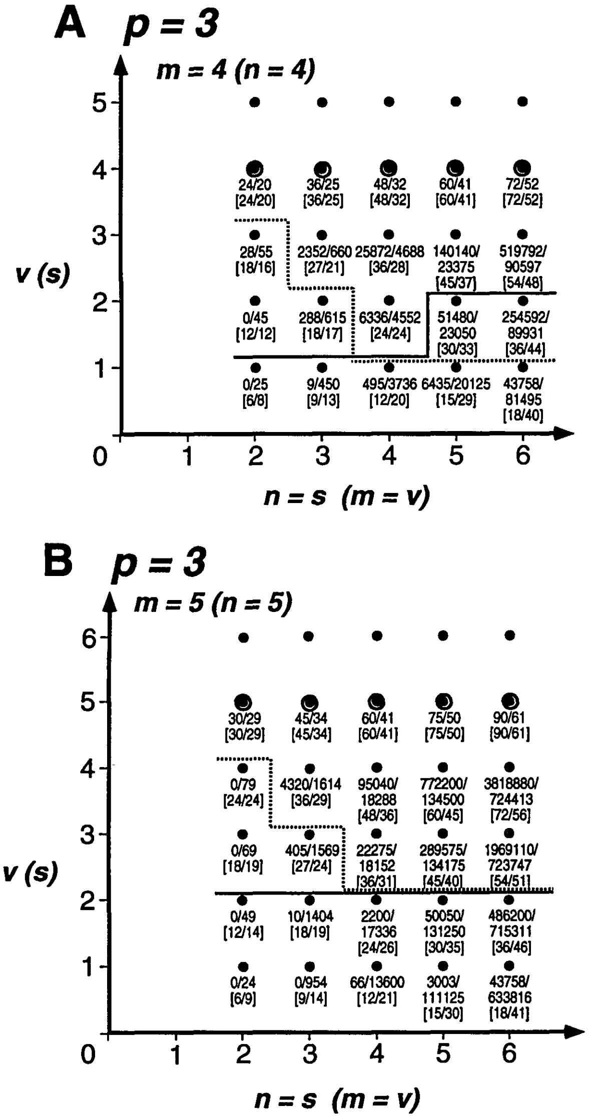


Fig. 2. Spectra of model check matrices for exemplary dichromatic bilinear models. Plotted are the ordered singular values of the model check matrices for exemplary bilinear models that combine the Smith–Pokorny protanope,[24] the CIE daylight basis for illumination,[25,30] and the Fourier basis for reflectance (see Table 2). The bottom two graphs pertain to Fig. 1A ($p = 2$, $m = 3$) and are spectra for matrices of models with parameters (2 3 3 3 3) (bottom, with a kernel of dimension three) and (2 3 4 3 4) (second from bottom, with a kernel of dimension one). To pass this model check, a matrix must have a kernel of dimension one, so that the results indicate failure for (2 3 3 3 3) and success for (2 3 4 3 4). The top two spectra pertain to Fig. 1B ($p = 2$, $m = 4$) and are from model check matrices for models with parameters (2 4 4 4 4) (second to top), with a kernel of dimension four, so failing the check, and (2 4 5 4 5) (top), with a kernel of dimension one, so passing the check.



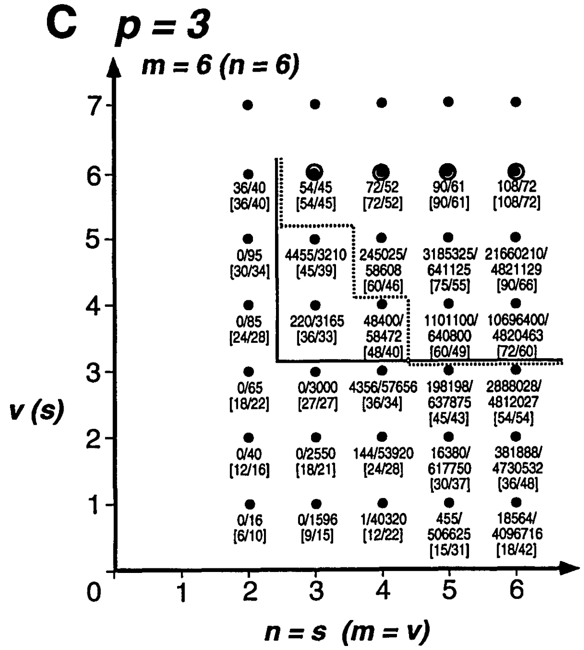


Fig. 3. Results for trichromatic bilinear models: A, the case of four-dimensional models of illumination ($p = 3$, $m = 4$, or by transposition $p = 3$, $n = 4$); B, the case of five-dimensional models of illumination ($p = 3$, $m = 5$, or by transposition $p = 3$, $n = 5$); C, the case of six-dimensional models of illumination ($p = 3$, $m = 6$, or by transposition $p = 3$, $n = 6$). The circles mark problems that are shown by the model-check algorithm to support perfect recovery procedures. See the caption for Fig. 1 and the text for further details.

B. Trichromacy

Figures 3A, 3B, and 3C show results for trichromatic problems of form $(p\ m\ n\ v\ s) = (3\ 4\ n\ v\ s)$, $(3\ 5\ n\ v\ s)$, and $(3\ 6\ n\ v\ s)$, respectively. With trichromatic systems one encounters feasible problems for which both the number p of photoreceptoral types and the number v of views are less than the dimension m of the illumination model. Beneath the points marking such problems are shown the numbers of equations and unknowns involved in the first nonlinear model check algorithm described elsewhere by us in format E_1/U_1 [Eqs. (A5) and (A11) of Ref. 23]. The dotted lines divide cases for which model checks are possible, in principle, from those that are not. The first model check involves systems of equations that are prohibitively large. The number Q of quantum-catch data and the number D of descriptors to be recovered are shown in bracketed format [Q/D] beneath all points.

Figure 3A shows results for problems with a four-dimensional model of illumination and reflectance model dimension n less than or equal to six. We have results only for problems in the top row, for which the model check algorithm of Subsection 4.B is applicable. The positive results for the problems (3 4 2 4 2) and (3 4 3 4 3) follow from transposition of the positive results[4] for (3 2 4 2 4) and (3 3 4 3 4), respectively. The model check algorithm shows that there are bilinear models with parameters (3 4 4 4 4), (3 4 5 4 5), and (3 4 6 4 6) that work perfectly. Exemplary spectra for these three problems are shown in Fig. 4. The bottom curves show that the kernels of model check matrices of exemplary bilinear models with parameters (3 4 4 4 4), (3 4 5 4 5), and (3 4 6 4 6) have dimension one, so that the models pass the check.

In Figure 3B are shown results for trichromatic problems with a five-dimensional model of illumination. The positive results for the problems with parameters $(p\ m\ n\ v\ s)$ = (3 5 2 5 2), (3 5 3 5 3), and (3 5 4 5 4) follow from transposition of the positive results for (3 2 5 2 5) and (3 3 5 3 5), found in Ref. 4, and for (3 4 5 4 5), shown in Fig. 3A. Continuing toward the right of Fig. 3B, the model check algorithm shows that there are bilinear models with parameters (3 5 5 5 5) and (3 5 6 5 6) that work perfectly. The middle two curves in Fig. 4 present exemplary spectra for these two problems.

Results for trichromatic problems with a six-

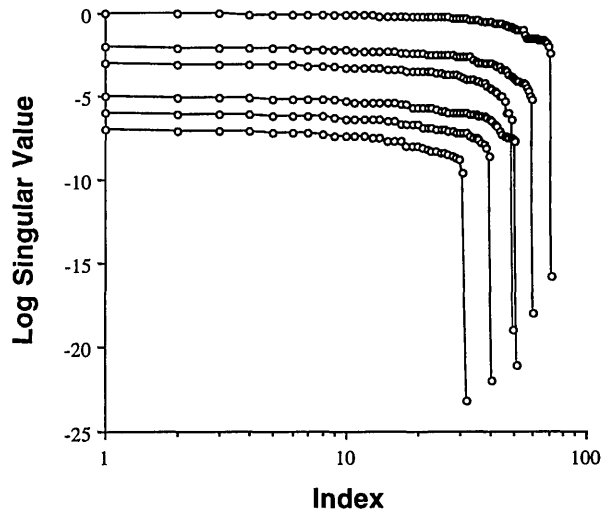


Fig. 4. Spectra of model check matrices for exemplary trichromatic bilinear models. The Smith–Pokorny[24] trichromat and the CIE daylight basis[25,30] were used in combination with Fourier reflectance models. The spectra shown are, from bottom to top, for the problems (3 4 4 4 4), (3 4 5 4 5), (3 4 6 4 6), (3 5 5 5 5), (3 5 6 5 6), and (3 6 6 6 6). The spectra show that the corresponding model check matrices had one-dimensional kernels, so passing the check. See the text for further discussion.

dimensional model of illumination are shown in Fig. 3C. The positive results for (3 6 3 6 3), (3 6 4 6 4), and (3 6 5 6 5) follow from transposition of positive results for (3 3 6 3 6), found in Ref. 4, for (3 4 6 4 6), shown in Fig. 3A, and for (3 5 6 5 6), shown in Fig. 3B. The model checks for (3 6 6 6 6) are successful; the top curve in Fig. 4 depicts an exemplary spectrum for this problem.

There are only three problems in Fig. 3 for which the second nonlinear model check algorithm described by us elsewhere[23] is possible. These problems meet two necessary conditions: (1) that the dimension n of the reflectance model equal or exceed the dimension m of the illumination model and (2) that the number E_2 of equations equal or exceed the number U_2 of unknowns [Eq. (B8) of Ref. 23]. The problem (3 4 4 3 4) provides a system of size $E_2/U_2 = 168/120$, the problem (3 5 5 4 5) provides a system of size $E_2/U_2 = 450/300$, and the problem (3 6 6 5 6) provides a system of size $E_2/U_2 = 990/630$. We have not implemented this model check algorithm.

C. Capacity of a Trichromatic Bilinear System

There are infinitely many choices of parameters that define feasible color constancy problems. In particular, all problems of the form $(p\ m\ n\ v\ s) = (3\ c\ c\ c\ c)$, c a positive integer, are feasible. In Fig. 3 we indicate the success of schemes with parameters (3 4 4 4 4), (3 5 5 5 5), and (3 6 6 6 6); similar results for (3 2 2 2 2) and (3 3 3 3 3) were found in earlier work.[4,5] One naturally wonders whether there are bilinear models with parameters $(3\ c\ c\ c\ c)$ for arbitrarily high c that provide unique recovery.

Figure 5 provides some numerical evidence. The figure shows the spectra of singular values for the model check matrices of exemplary bilinear models with parameters $(3\ c\ c\ c\ c)$ for $3 \leq c \leq 31$, c odd. The exemplary models comprise the Smith–Pokorny[24] fundamentals and the Fourier models for reflectance and illumination; the latter are of dimension c and have their highest frequency components at frequency $(c - 1)/2$. The upper limit 31 stems from our choice to approximate functions of wavelength using the 31-dimensional vectors that arise in sampling the interval 400–700 nm at 10-nm intervals.[4] The number v of views is identical to the dimension m for illumination, viz., $v = m = c$, so that the linear model check algorithm can be applied. Figure 5 shows that each of the exemplary models gives rise to a model check matrix with a kernel of dimension one: each model passes the check.

The numerical results lead to the following conjecture: a trichromatic visual system can recover arbitrarily high numbers of descriptors per illuminant and reflectance when provided adequate information. We now prove this claim for problems with parameters of the form $(3\ c\ c\ c\ c)$, to which we apply the analysis based on Schur's lemma[32] that was introduced by Iverson and D'Zmura.[5]

We define the three $c \times c$ bilinear model matrices $\boldsymbol{\beta}_k$, $k = 1, 2, 3$, with entries $(\boldsymbol{\beta}_k)_{ji} = (\mathbf{B}_j)_{ki}$ [see Eq. (1)], so providing one bilinear model matrix per photoreceptoral spectral sensitivity.[5] We use these matrices to rewrite Eq. (26) as follows [Eq. (4) of Ref. 5]:

$$\boldsymbol{\Delta}_k = \mathbf{R}\boldsymbol{\beta}_k\mathbf{A}, \qquad k = 1, 2, 3, \tag{40}$$

in which the data matrices $\boldsymbol{\Delta}_k$ have entries q_{twk} [see Eq. (2)] and dimension $n \times v$ and the matrix of reflectances $\mathbf{R}$ has entries r_{tj} and dimension $n \times n$.

Suppose now that surfaces with reflectance descriptor matrix $\mathbf{R}$ lit by sources with descriptor matrix $\mathbf{A}$ give rise to quantum catches that are identical to those produced

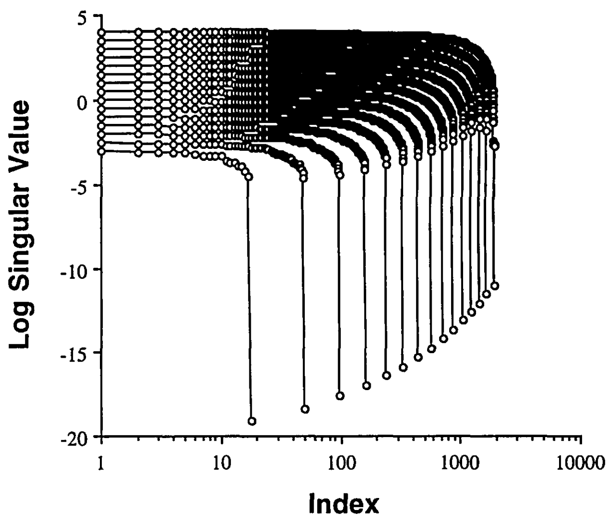


Fig. 5. Spectra of model check matrices for exemplary trichromatic bilinear models with parameters $(3\ c\ c\ c\ c)$ for $3 \leq c \leq 31$, c odd. Plotted on a log axis are the ordered singular values of the model check matrices for these models. We scaled the spectra to stagger the maximal singular values along the vertical axis at half-log-unit intervals. The parameters of the models whose spectra are shown are, from bottom to top, (3 3 3 3 3), (3 5 5 5 5), (3 7 7 7 7), (3 9 9 9 9), (3 11 11 11 11), (3 13 13 13 13), (3 15 15 15 15), (3 17 17 17 17), (3 19 19 19 19), (3 21 21 21 21), (3 23 23 23 23), (3 25 25 25 25), (3 27 27 27 27), (3 29 29 29 29), and (3 31 31 31 31). Each has a kernel of dimension one and so passes the necessary and sufficient model check. See the text for further discussion.

by surfaces with reflectance descriptor matrix $\mathbf{S}$ lit by sources with descriptor matrix $\mathbf{Z}$:

$$\mathbf{R}\boldsymbol{\beta}_k\mathbf{A} = \mathbf{S}\boldsymbol{\beta}_k\mathbf{Z}, \qquad k = 1, 2, 3. \tag{41}$$

We want to identify bilinear models for which the only possible solution to Eq. (41) is that where $\mathbf{S} = \kappa\mathbf{R}$ and $\mathbf{A} = (1/\kappa)\mathbf{Z}$ for some constant κ. Note that the $c \times c$ matrices $\mathbf{S}$ and $\mathbf{A}$ are invertible, by hypothesis, so leading to

$$\mathbf{E}\boldsymbol{\beta}_k = \boldsymbol{\beta}_k\mathbf{H}, \qquad k = 1, 2, 3, \tag{42}$$

where $\mathbf{E} = \mathbf{S}^{-1}\mathbf{R}$ and $\mathbf{H} = \mathbf{Z}\mathbf{A}^{-1}$. For problems of the form $(3\ c\ c\ c\ c)$ the $c \times c$ matrices $\boldsymbol{\beta}_k$, $k = 1, 2, 3$, are invertible, producing

$$\boldsymbol{\beta}_k^{-1}\mathbf{E}\boldsymbol{\beta}_k = \mathbf{H}, \qquad k = 1, 2, 3. \tag{43}$$

It follows at once that

$$\boldsymbol{\beta}_k^{-1}\mathbf{E}\boldsymbol{\beta}_k = \boldsymbol{\beta}_l^{-1}\mathbf{E}\boldsymbol{\beta}_l, \qquad k, l = 1, 2, 3. \tag{44}$$

Let us define the associated model matrices $\mathsf{G}_{kl} = \boldsymbol{\beta}_k\boldsymbol{\beta}_l^{-1}$, in terms of which Eq. (44) becomes

$$\mathbf{E}\mathsf{G}_{kl} = \mathsf{G}_{kl}\mathbf{E}, \qquad k, l = 1, 2, 3, \tag{45}$$

and we see that $\mathbf{E}$ commutes with each of the associated model matrices. Following Iverson and D'Zmura,[5] it suffices to consider two of these matrices, say, G_{21} and G_{31}. By Eq. (45),

$$[\mathbf{E}, \mathsf{G}_{21}] = \mathbf{0} = [\mathbf{E}, \mathsf{G}_{31}], \tag{46}$$

where the symbol $[\mathbf{A}, \mathbf{B}]$ denotes the commutator $\mathbf{AB}-\mathbf{BA}$.

By Schur's lemma,[5,32] if G_{21} and G_{31} share no common nontrivial invariant subspace, then the matrix $\mathbf{E}$ is a multiple of the identity matrix; i.e., the two sets of surface reflectances are related by a single positive scalar, and the two sets of light sources are related by the reciprocal of that scalar. It thus suffices to show that one can always find matrices G_{21} and G_{31}, for arbitrarily high c, that have no common nontrivial invariant subspace.

Let us choose G_{21} to possess distinct eigenvalues. It follows that G_{21} can be diagonalized, and without loss of generality we take

$$\mathsf{G}_{21} = \begin{bmatrix} \mu_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mu_c \end{bmatrix}. \tag{47}$$

It now follows from Eq. (46) that matrix $\mathbf{E}$ is diagonal, because the components $[\mathbf{E}, \mathsf{G}_{21}]_{ij}$ of the commutator are $e_{ij}(\mu_j - \mu_i)$; and, because the eigenvalues of G_{21} are distinct, the off-diagonal components of $\mathbf{E}$, namely e_{ij} for $i \neq j$, must vanish.

Consider now the components $[\mathbf{E}, \mathsf{G}_{31}]_{ij}$ of the commutator $[\mathbf{E}, \mathsf{G}_{31}]$; these are $(e_{ii} - e_{jj})\mathsf{g}_{ij}$, where g_{ij} are the components of G_{31}. Since these expressions vanish for all i, j by Eq. (46), the simplest way to ensure that $e_{ii} = e_{jj}$ for all $i, j = 1, 2, \ldots, c$, is to construct G_{31} so that $\mathsf{g}_{ij} \neq 0$ for some i and all j. This can always be done.

For color-constancy problems with parameters $(3\ c\ c\ c\ c)$, for arbitrarily large c, we can thus always construct bilinear models with modified gamma matrices that share no common nontrivial invariant subspace. By Schur's lemma, the matrix $\mathbf{E}$ is a scalar multiple of the identity, and so the bilinear model provides a one-to-one relationship between lit surfaces and quantum-catch data, up to an arbitrary positive scalar.

Note that the ability to specify associated model matrices G_{21} and G_{31} that force the scaling solution is tantamount to specifying bilinear model matrices $\boldsymbol{\beta}_k$, $k = 1, 2, 3$, that provide perfect recovery. Perhaps the simplest of many ways to construct model matrices $\boldsymbol{\beta}_k$, $k = 1, 2, 3$, given suitable G_{21} and G_{31}, is to pick an arbitrary invertible $\boldsymbol{\beta}_1$ and set $\boldsymbol{\beta}_2 = \mathsf{G}_{21}\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_3 = \mathsf{G}_{31}\boldsymbol{\beta}_1$.

D. Capacity of a Polychromatic Bilinear System

The result of Subsection 5.C shows that there are trichromatic systems that can recover c descriptors, $c \geq 2$. It must be the case that there are systems with greater numbers of photoreceptoral types that can also recover c descriptors, $c \geq 2$. We can easily reduce such a system to a trichromatic system, for which we know recovery is possible, simply by ignoring the responses of the excess photoreceptoral types. This argument is equivalent to one that applies entailment (a) of Ref. 4: $(3\ c\ c\ c\ c) \Rightarrow (p\ c\ c\ c\ c)$, $p > 3$. We conclude that there are p-chromatic visual systems that can recover c descriptors for both illuminants and reflectances, for $p \geq 3$ and $c \geq 2$.

In contrast, all bilinear models with parameters of the form $(2\ c\ c\ c\ c)$ produce recovery algorithms that fail totally, which follows by extending the proof[4] of the total failure of the special case $(2\ 2\ 2\ 2\ 2)$.

We investigated further chains of problems of this sort. We performed successful model checks of all problems of the form $(2\ c\ c+1\ c\ c+1)$, for $3 \leq c \leq 29$. The success of these model checks implies success, through entailment (a) of Ref. 4, for problems of form $(p\ c\ c+1\ c\ c+1)$ for $p \geq 2$ and $3 \leq c \leq 29$ (and their transposes). Although we have not proved the existence of perfect recovery algorithms for arbitrarily high c in these chains, we are confident that this is the case.

The successful results of the further model checks, performed on models with illuminant and surface parameters limited to a maximum value of 31, suggest that there are many chains, including $(p\ c\ c+2\ c\ c+2)$, $(p\ c\ c+3\ c\ c+3)$, $(p\ c\ c+4\ c\ c+4)$, and $(p\ c\ c+5\ c\ c+5)$, for $p \geq 3$ and $c \geq 3$, for which recovery works perfectly for a wide variety of bilinear models.

6. DISCUSSION

Our goal has been to describe and analyze a general algorithm that recovers spectral descriptions for lights and surfaces simultaneously, using linear methods. This simultaneous recovery is possible whenever the number of surfaces equals or exceeds the number of reflectance descriptors to be recovered per surface. The general linear recovery procedure can be used for all color constancy problems amenable to two-stage linear recovery.[3,4] General linear recovery has a broader scope because the number of photoreceptoral types need not equal or exceed the dimension of the model for illumination. The problems that are not amenable to general linear recovery form a relatively small class that are inherently nonlinear.[21]

Like the linear model check algorithm for two-stage linear recovery,[3,4] that for general linear recovery works well to classify bilinear models. Yet the nonlinear model check algorithm for general linear recovery, met in cases in which the number v of views is less than the dimension m for illumination, leads to linearized systems of equations that are usually too large for us to treat numerically.

Analytic methods suffice to show that there are infinite chains $(p\ c\ c\ c\ c)$, $p \geq 3$ and $c \geq 2$, of color-constancy problems for which general linear recovery works perfectly. This result supports the intuition that polychromatic systems can use reflected lights to determine spectral descriptions of arbitrarily high dimension when provided adequate information. Of course, this does not imply that an arbitrary polychromatic visual system has this property. Yet the successful results of our numerical tests through $c = 31$, with the empirically motivated trichromatic systems of Table 2, suggest that standard trichromatic systems have a very high capacity for recovering spectral descriptions.

Numerical results suggest that there are a number of chains, including $(p\ c\ c+1\ c\ c+1)$ for $p \geq 2$ and $(p\ c\ c+2\ c\ c+2)$, $(p\ c\ c+3\ c\ c+3)$, $(p\ c\ c+4\ c\ c+4)$, and $(p\ c\ c+5\ c\ c+5)$ for $p \geq 3$, for which recovery works perfectly for a variety of bilinear models. Further chains are feasible if the base number of photoreceptoral types is increased [for example, if $p \geq 4$, the chain $(p\ c\ c+6\ c\ c+6)$ is feasible], and we conjecture that all such chains provide perfect recovery schemes that can work to arbitrarily high accuracy.

ACKNOWLEDGMENTS

This research was supported by National Eye Institute grant EY10014 to M. D'Zmura and by National Science Foundation grant DIR-9014278 to the Institute for Mathematical Behavioral Sciences, University of California, Irvine, R. D. Luce, director.

REFERENCES

1. L. T. Maloney and B. A. Wandell, "Color constancy: a method for recovering surface spectral reflectance," J. Opt. Soc. Am. A **3**, 29–33 (1986).
2. M. D'Zmura, "Color constancy: surface color from changing illumination," J. Opt. Soc. Am. A **9**, 490–493 (1992).
3. M. D'Zmura and G. Iverson, "Color constancy. I. Basic theory of two-stage linear recovery of spectral descriptions for lights and surfaces," J. Opt. Soc. Am. A **10**, 2148–2165 (1993).
4. M. D'Zmura and G. Iverson, "Color constancy. II. Results for two-stage linear recovery of spectral descriptions for lights and surfaces," J. Opt. Soc. Am. A **10**, 2166–2180 (1993).
5. G. Iverson and M. D'Zmura, "Criteria for color constancy in trichromatic bilinear models," J. Opt. Soc. Am. A **11**, 1970–1975 (1994).
6. D. Katz, *The World of Color,* R. B. MacLeod and C. W. Fox, trans. (Kegan Paul, Trench, Trubner, London, 1935).
7. J. Beck, *Surface Color Perception* (Cornell U. Press, Ithaca, N.Y., 1972).
8. E. H. Land, "Recent advances in retinex theory," Vision Res. **26**, 7–21 (1986).
9. P. Sällström, "Colour and physics: some remarks concerning the physical aspects of human colour vision," University of Stockholm Institute of Physics Rep. 73-09 (University of Stockholm, Stockholm, 1973).
10. M. H. Brill, "A device performing illuminant-invariant assessment of chromatic relations," J. Theor. Biol. **71,** 473–478 (1978).
11. M. H. Brill, "Further features of the illuminant-invariant trichromatic photosensor," J. Theor. Biol. **78**, 305–308 (1979).
12. G. Buchsbaum, "A spatial processor model for object colour perception," J. Franklin Inst. **310,** 1–26 (1980).
13. L. T. Maloney, "Computational approaches to color constancy," Stanford Applied Psychology Laboratory Tech. Rep. 1985-01 (Stanford University, Palo Alto, Calif., 1985).
14. A. Hurlbert, "Formal connections between lightness algorithms," J. Opt. Soc. Am. A **3**, 1684–1693 (1986).
15. M. D'Zmura and P. Lennie, "Mechanisms of color constancy," J. Opt. Soc. Am. A **3,** 1662–1672 (1986).
16. H.-C. Lee, "Method for computing the scene-illuminant chromaticity from specular highlights," J. Opt. Soc. Am. A **3,** 1694–1699 (1986).
17. D. H. Brainard, B. A. Wandell, and W. B. Cowan, "Black light: how sensors filter spectral variation of the illuminant," IEEE Trans. Biomed. Eng. **36**, 140–149 (1989).
18. J. Rubner and K. Schulten, "A regularized approach to color constancy," Biol. Cybern. **61**, 29–36 (1989).
19. D. A. Forsyth, "A novel algorithm for color constancy," Int. J. Comput. Vis. **5**, 5–36 (1990).
20. M. S. Drew and B. V. Funt, "Variational approach to interreflection in color images," J. Opt. Soc. Am. A **9**, 1255–1265 (1992).
21. M. D'Zmura and G. Iverson, "Color constancy: feasibility and recovery," Invest. Ophthalmol. Vis. Sci. Suppl. **34,** 748 (1993).
22. W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes in C. The Art of Scientific Computing* (Cambridge U. Press, New York, 1988).
23. M. D'Zmura and G. Iverson, "Color constancy. III. General linear recovery of spectral descriptions for lights and surfaces," Institute for Mathematical Behavioral Sciences Tech. Rep. MBS 93-38 (University of California, Irvine, Irvine, Calif., 1993).
24. V. C. Smith and J. Pokorny, "Spectral sensitivity of the foveal cone photopigments between 400 and 500 nm," Vision Res. **15**, 161–171 (1975).
25. D. B. Judd, D. L. MacAdam, and G. Wyszecki, "Spectral distribution of typical daylight as a function of correlated color temperature," J. Opt. Soc. Am. **54**, 1031–1040 (1964).
26. J. Cohen, "Dependency of the spectral reflectance curves of the Munsell color chips," Psychonom. Sci. **1**, 369–370 (1964).
27. E. R. Dixon, "Spectral distribution of Australian daylight," J. Opt. Soc. Am. **68**, 437–450 (1978).
28. J. P. S. Parkkinen, J. Hallikainen, and T. Jaaskelainen, "Characteristic spectra of Munsell colors," J. Opt. Soc. Am. A **6**, 318–322 (1989).
29. L. M. Hurvich and D. Jameson, "Some quantitative aspects of an opponent-color theory. II. Brightness, saturation, and hue in normal and dichromatic vision," J. Opt. Soc. Am. **45**, 602–616 (1955).
30. G. Wyszecki and W. S. Stiles, *Color Science. Concepts and Methods, Quantitative Data and Formulas,* 2nd ed. (Wiley, New York, 1982).
31. A. Stockman, D. I. A. MacLeod, and N. E. Johnson, "Spectral sensitivities of the human cones," J. Opt. Soc. Am. A **10,** 2491–2521 (1993).
32. S. Lang, *Linear Algebra,* 2nd ed. (Addison-Wesley, Reading, Mass., 1971).