Minimum Wage and Employer Variety

Appendix — For Online Publication

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A Supporting Tables

Table A-1: Pair-approach estimation of minimum wage responses of U.S. establishment counts with additional control, 1990–2016

	Sixteen industries (sorted by 1990 earnings per worker)					
Industry →	**Overall**	**1**	**2**	**3**	**4**	**5**
ln(minimum wage)	-0.091*	-0.183	-0.166	-0.097**	-0.044	0.063
	(0.048)	(0.112)	(0.111)	(0.040)	(0.046)	(0.148)
ln(population)	0.866***	0.789***	0.336	0.242**	0.279**	0.599*
	(0.083)	(0.155)	(0.249)	(0.110)	(0.122)	(0.340)
ln(establishments^{-})		0.322***	0.314**	0.442***	0.504***	0.519***
		(0.109)	(0.150)	(0.072)	(0.076)	(0.174)
Observations	8,134	8,134	7,798	8,134	8,132	7,920
Industry →	**6**	**7**	**8**	**9**	**10**	**11**
ln(minimum wage)	0.080	-0.128	-0.192	-0.021	-0.049	0.028
	(0.139)	(0.093)	(0.141)	(0.093)	(0.095)	(0.078)
ln(population)	0.480*	0.676***	0.778	0.276	0.301	0.218
	(0.248)	(0.149)	(0.624)	(0.206)	(0.277)	(0.138)
ln(establishments^{-})	0.847***	0.353**	0.528**	1.026***	0.647***	0.743***
	(0.183)	(0.139)	(0.250)	(0.135)	(0.231)	(0.121)
Observations	7,962	8,128	7,592	8,090	8,128	8,128
Industry →	**12**	**13**	**14**	**15**	**16**	
ln(minimum wage)	-0.001	0.075	-0.090	0.151	-1.497***	
	(0.084)	(0.104)	(0.102)	(0.178)	(0.415)	
ln(population)	0.316**	0.176	-0.269	0.309	0.534	
	(0.137)	(0.227)	(0.211)	(0.552)	(0.859)	
ln(establishments^{-})	0.501***	0.513***	0.705***	0.373	1.092**	
	(0.142)	(0.152)	(0.147)	(0.280)	(0.421)	
Observations	8,094	8,112	7,808	8,014	7,022	

Notes: This table reports $\hat{\beta}$ and $\hat{\gamma}$ from the estimation of equation (1)—after also adding establishment counts in all other industries as control—for 16 industries and overall using yearly data from 1990 to 2016. All regressions include commuting zone–state fixed effects and pair–year fixed effects. Industries are sorted according to the 1990 earnings ranking of Table 1. Standard errors (in parentheses) are two-way clustered at the state and border segment levels. The coefficients are statistically significant at the *10%, **5%, or ***1% level.

Table A-2: Long-term minimum wage responses of U.S. establishment counts, 1990–2016

Industry ↓	$\hat{\beta}_2$	$\hat{\beta}_1$	$\hat{\beta}_0$	$\hat{\beta}_{-1}$	$\hat{\beta}_{-2}$	$\hat{\beta}_{-3}$	$\hat{\beta}_{-4}$
Overall	-0.045	-0.043	-0.056	-0.090	-0.121*	-0.153**	-0.224**
	(0.030)	(0.036)	(0.049)	(0.063)	(0.071)	(0.075)	(0.095)
1	-0.186**	-0.160	-0.215*	-0.321*	-0.403**	-0.502**	-0.662**
	(0.082)	(0.096)	(0.120)	(0.163)	(0.191)	(0.219)	(0.272)
2	-0.113	-0.028	-0.004	-0.116	-0.113	-0.087	-0.327
	(0.126)	(0.126)	(0.178)	(0.187)	(0.214)	(0.243)	(0.242)
3	-0.075*	-0.088**	-0.140**	-0.154**	-0.215***	-0.250***	-0.298***
	(0.039)	(0.044)	(0.058)	(0.075)	(0.078)	(0.078)	(0.092)
4	-0.029	-0.039	-0.051	-0.082	-0.063	-0.070	-0.120
	(0.043)	(0.052)	(0.050)	(0.058)	(0.070)	(0.076)	(0.107)
5	0.189	0.016	-0.177	-0.154	-0.270	-0.257	-0.420
	(0.114)	(0.139)	(0.191)	(0.210)	(0.238)	(0.286)	(0.364)
6	-0.003	-0.085	-0.099	-0.147	-0.024	-0.001	-0.237
	(0.115)	(0.129)	(0.150)	(0.182)	(0.204)	(0.239)	(0.328)
7	-0.006	-0.028	-0.061	-0.109	-0.166	-0.242	-0.402*
	(0.063)	(0.068)	(0.086)	(0.125)	(0.146)	(0.168)	(0.232)
8	-0.320**	-0.434***	-0.352**	-0.426**	-0.557**	-0.730**	-0.614**
	(0.145)	(0.147)	(0.163)	(0.196)	(0.263)	(0.286)	(0.280)
9	0.006	0.058	0.010	-0.078	-0.103	-0.139	-0.178
	(0.072)	(0.094)	(0.112)	(0.139)	(0.165)	(0.203)	(0.256)
10	-0.041	-0.022	-0.048	-0.105	-0.129	-0.191	-0.218
	(0.079)	(0.090)	(0.119)	(0.132)	(0.154)	(0.160)	(0.215)
11	0.003	0.007	0.034	0.030	-0.006	-0.014	-0.133
	(0.053)	(0.054)	(0.064)	(0.086)	(0.096)	(0.103)	(0.125)
12	-0.078	-0.175**	-0.162**	-0.134	-0.130	-0.123	-0.151
	(0.065)	(0.071)	(0.080)	(0.098)	(0.110)	(0.127)	(0.148)
13	0.020	0.074	0.067	0.123	0.143	0.119	0.123
	(0.072)	(0.093)	(0.098)	(0.108)	(0.123)	(0.134)	(0.173)
14	-0.062	-0.126	-0.193*	-0.165	-0.186	-0.219	-0.324
	(0.081)	(0.100)	(0.107)	(0.133)	(0.160)	(0.186)	(0.217)
15	0.007	0.206	0.204	0.178	0.230	0.258	-0.005
	(0.125)	(0.136)	(0.151)	(0.201)	(0.195)	(0.229)	(0.214)
16	-1.145***	-0.598	-0.503	-1.090***	-1.290***	-1.839***	-2.593***
	(0.379)	(0.422)	(0.418)	(0.372)	(0.441)	(0.532)	(0.631)

Notes: This table reports $\hat{\beta}_k$, for $k \in \{2, 1, 0, -1, -2, -3, -4\}$ from the estimation of equation (2) for 16 industries and overall using yearly data from 1990 to 2016. All regressions include the log of working-age population as control, as well as commuting zone–state fixed effects and pair–year fixed effects. Industries are sorted according to the 1990 earnings ranking of Table 1. Standard errors (in parentheses) are two-way clustered at the state and border segment levels. The coefficients are statistically significant at the *10%, **5%, or ***1% level.

Table A-3: Pair-approach estimation of minimum wage responses of U.S. employment, 1990–2016

	Sixteen industries (sorted by 1990 earnings per worker)					
Industry →	**Overall**	**1**	**2**	**3**	**4**	**5**
ln(minimum wage)	-0.204**	-0.273**	0.140	-0.119*	-0.048	-0.310
	(0.084)	(0.119)	(0.349)	(0.067)	(0.085)	(0.340)
ln(population)	0.999***	1.093***	1.054	0.753***	0.707***	0.720*
	(0.098)	(0.179)	(0.705)	(0.102)	(0.171)	(0.376)
Observations	8,134	8,134	7,798	8,134	8,132	7,920
Industry →	**6**	**7**	**8**	**9**	**10**	**11**
ln(minimum wage)	-0.024	-0.114	-0.036	-0.221	-0.215	-0.132
	(0.315)	(0.109)	(0.369)	(0.200)	(0.151)	(0.111)
ln(population)	0.801*	0.899***	2.201**	1.574***	0.959***	0.894***
	(0.399)	(0.233)	(0.824)	(0.425)	(0.295)	(0.220)
Observations	7,962	8,128	7,592	8,090	8,128	8,128
Industry →	**12**	**13**	**14**	**15**	**16**	
ln(minimum wage)	-0.057	-0.470*	-0.329	0.104	-2.491***	
	(0.144)	(0.267)	(0.319)	(0.238)	(0.587)	
ln(population)	1.422***	0.411	-0.167	0.999*	2.139**	
	(0.201)	(0.294)	(0.447)	(0.511)	(0.922)	
Observations	8,094	8,112	7,808	8,014	7,022	

Notes: This table reports $\hat{\beta}$ and $\hat{\gamma}$ from the estimation of equation (1) for 16 industries and overall using 1990-2016 yearly data, but using log employment instead of log establishment counts as the dependent variable. All regressions include commuting zone–state fixed effects and pair–year fixed effects. Industries are sorted according to the 1990 earnings ranking of Table 1. Standard errors (in parentheses) are two-way clustered at the state and border segment levels. The coefficients are statistically significant at the *10%, **5%, or ***1% level.

Table A-4: Pair-approach estimation of minimum wage responses of U.S. earnings per worker, 1990–2016

	Sixteen industries (sorted by 1990 earnings per worker)					
Industry →	**Overall**	**1**	**2**	**3**	**4**	**5**
ln(minimum wage)	0.017	0.164***	0.144	0.039	-0.012	0.063
	(0.055)	(0.055)	(0.153)	(0.043)	(0.089)	(0.170)
ln(population)	0.093	0.096	0.100	-0.169	0.095	-0.161
	(0.083)	(0.087)	(0.195)	(0.111)	(0.131)	(0.250)
Observations	8,134	8,134	7,798	8,134	8,132	7,920
Industry →	**6**	**7**	**8**	**9**	**10**	**11**
ln(minimum wage)	0.094	0.126*	-0.118	0.070	-0.037	-0.088
	(0.107)	(0.069)	(0.107)	(0.093)	(0.077)	(0.058)
ln(population)	-0.222*	0.107	-0.006	0.040	0.144	0.099
	(0.127)	(0.122)	(0.119)	(0.163)	(0.101)	(0.075)
Observations	7,962	8,128	7,592	8,090	8,128	8,128
Industry →	**12**	**13**	**14**	**15**	**16**	
ln(minimum wage)	0.043	0.094	-0.127	0.010	0.086	
	(0.089)	(0.077)	(0.098)	(0.126)	(0.166)	
ln(population)	-0.022	0.113	0.055	0.035	-0.434	
	(0.194)	(0.106)	(0.137)	(0.144)	(0.337)	
Observations	8,094	8,112	7,808	8,014	7,022	

Notes: This table reports $\hat{\beta}$ and $\hat{\gamma}$ from the estimation of equation (1) for 16 industries and overall using 1990-2016 yearly data, but using log earnings per worker instead of log establishment counts as the dependent variable. All regressions include commuting zone–state fixed effects and pair–year fixed effects. Industries are sorted according to the 1990 earnings ranking of Table 1. Standard errors (in parentheses) are two-way clustered at the state and border segment levels. The coefficients are statistically significant at the *10%, **5%, or ***1% level.

Table A-5: Long-term minimum wage responses of U.S. employment, 1990–2016

Industry ↓	$\hat{\beta}_2$	$\hat{\beta}_1$	$\hat{\beta}_0$	$\hat{\beta}_{-1}$	$\hat{\beta}_{-2}$	$\hat{\beta}_{-3}$	$\hat{\beta}_{-4}$
Overall	-0.080*	-0.118*	-0.167**	-0.271**	-0.316**	-0.400***	-0.418**
	(0.045)	(0.060)	(0.079)	(0.107)	(0.131)	(0.148)	(0.188)
1	-0.093	-0.156	-0.185*	-0.354***	-0.547***	-0.577***	-0.719***
	(0.084)	(0.107)	(0.098)	(0.125)	(0.156)	(0.162)	(0.188)
2	-0.146	-0.351	-0.019	0.083	0.172	0.152	0.340
	(0.207)	(0.269)	(0.329)	(0.436)	(0.497)	(0.584)	(0.664)
3	-0.079	-0.075	-0.069	-0.117	-0.126	-0.222**	-0.200
	(0.062)	(0.063)	(0.071)	(0.092)	(0.100)	(0.108)	(0.137)
4	0.011	0.011	0.034	0.017	-0.002	0.031	-0.108
	(0.075)	(0.084)	(0.098)	(0.107)	(0.105)	(0.126)	(0.135)
5	0.045	-0.154	-0.299	-0.122	-0.444	-0.395	-0.924
	(0.329)	(0.394)	(0.495)	(0.606)	(0.668)	(0.645)	(0.889)
6	-0.340*	-0.096	-0.176	0.023	0.129	0.075	0.486
	(0.174)	(0.191)	(0.209)	(0.264)	(0.298)	(0.377)	(0.518)
7	-0.123	-0.097	-0.245**	-0.164	-0.213	-0.242	-0.308
	(0.079)	(0.083)	(0.118)	(0.125)	(0.138)	(0.150)	(0.205)
8	-0.261	-0.007	0.014	0.022	-0.023	0.378	0.360
	(0.344)	(0.348)	(0.329)	(0.415)	(0.499)	(0.509)	(0.647)
9	0.054	0.049	0.180	-0.137	-0.141	-0.342	-0.193
	(0.157)	(0.182)	(0.177)	(0.245)	(0.271)	(0.311)	(0.434)
10	0.005	-0.128	-0.046	-0.365*	-0.125	-0.440*	-0.312
	(0.094)	(0.110)	(0.151)	(0.199)	(0.239)	(0.228)	(0.275)
11	-0.037	-0.095	-0.091	-0.079	-0.209	-0.215	-0.384*
	(0.094)	(0.120)	(0.131)	(0.156)	(0.178)	(0.182)	(0.210)
12	0.049	0.052	-0.090	-0.114	-0.050	-0.169	-0.276
	(0.113)	(0.137)	(0.157)	(0.205)	(0.193)	(0.197)	(0.208)
13	-0.433**	-0.474**	-0.570*	-0.630*	-0.659	-0.824*	-0.842
	(0.176)	(0.219)	(0.292)	(0.352)	(0.403)	(0.451)	(0.553)
14	0.147	-0.048	-0.290	-0.367	-0.575	-0.663	-0.626
	(0.162)	(0.172)	(0.232)	(0.346)	(0.452)	(0.555)	(0.797)
15	0.223	0.276	0.116	-0.039	-0.140	-0.245	-0.506
	(0.168)	(0.216)	(0.247)	(0.273)	(0.295)	(0.348)	(0.411)
16	-1.945***	-1.580*	-1.522*	-2.472***	-2.740***	-3.253***	-4.338***
	(0.663)	(0.849)	(0.900)	(0.861)	(0.968)	(1.029)	(1.067)

Notes: This table reports $\hat{\beta}_k$, for $k \in \{2, 1, 0, -1, -2, -3, -4\}$ from the estimation of equation (2) for 16 industries and overall using yearly data from 1990 to 2016, but using log employment instead of log establishment counts as the dependent variable. All regressions include the log of working-age population as control, as well as commuting zone–state fixed effects and pair–year fixed effects. Industries are sorted according to the 1990 earnings ranking of Table 1. Standard errors (in parentheses) are two-way clustered at the state and border segment levels. The coefficients are statistically significant at the *10%, **5%, or ***1% level.

Table A-6: Long-term minimum wage responses of U.S. earnings per worker, 1990–2016

Industry ↓	$\hat{\beta}_2$	$\hat{\beta}_1$	$\hat{\beta}_0$	$\hat{\beta}_{-1}$	$\hat{\beta}_{-2}$	$\hat{\beta}_{-3}$	$\hat{\beta}_{-4}$
Overall	-0.001	-0.027	-0.025	0.011	0.012	0.045	0.004
	(0.043)	(0.052)	(0.062)	(0.069)	(0.071)	(0.078)	(0.090)
1	0.165*	0.148	0.287**	0.345***	0.306***	0.257***	0.169*
	(0.096)	(0.110)	(0.112)	(0.121)	(0.086)	(0.094)	(0.086)
2	-0.017	0.238	0.367*	0.323	0.199	0.299	0.293
	(0.109)	(0.147)	(0.202)	(0.258)	(0.288)	(0.339)	(0.346)
3	-0.042	-0.067*	0.011	0.037	0.030	0.070	0.076
	(0.041)	(0.038)	(0.050)	(0.061)	(0.071)	(0.079)	(0.088)
4	-0.083	-0.182*	-0.050	-0.152	-0.060	-0.130	-0.107
	(0.074)	(0.095)	(0.107)	(0.122)	(0.120)	(0.136)	(0.173)
5	0.161	0.135	0.239	0.332	0.365	0.187	0.007
	(0.163)	(0.203)	(0.299)	(0.329)	(0.327)	(0.321)	(0.356)
6	-0.014	0.013	0.218	0.114	-0.004	0.201	-0.024
	(0.113)	(0.125)	(0.145)	(0.149)	(0.149)	(0.171)	(0.195)
7	0.045	-0.011	0.079	0.071	0.132	0.192*	0.209
	(0.065)	(0.062)	(0.075)	(0.080)	(0.089)	(0.105)	(0.129)
8	-0.002	-0.133	-0.225**	-0.229**	-0.243*	-0.297**	-0.325*
	(0.079)	(0.080)	(0.091)	(0.111)	(0.133)	(0.134)	(0.165)
9	-0.083	0.040	-0.019	0.171	0.063	0.346*	0.235
	(0.100)	(0.117)	(0.127)	(0.133)	(0.162)	(0.182)	(0.229)
10	0.036	-0.004	-0.076	-0.041	-0.142	-0.010	-0.124
	(0.082)	(0.088)	(0.116)	(0.118)	(0.120)	(0.130)	(0.158)
11	-0.081	-0.064	-0.115	-0.133	-0.099	-0.202**	-0.255**
	(0.062)	(0.064)	(0.073)	(0.093)	(0.092)	(0.095)	(0.111)
12	0.045	0.021	-0.043	0.091	-0.116	-0.018	0.039
	(0.075)	(0.078)	(0.091)	(0.103)	(0.125)	(0.111)	(0.137)
13	-0.059	-0.001	0.017	0.064	0.075	0.194	0.071
	(0.100)	(0.105)	(0.116)	(0.144)	(0.151)	(0.155)	(0.188)
14	-0.068	-0.163*	-0.134	-0.202	-0.195	-0.223	-0.246
	(0.066)	(0.082)	(0.104)	(0.143)	(0.182)	(0.190)	(0.234)
15	-0.187**	-0.084	-0.065	0.029	-0.110	0.033	-0.031
	(0.082)	(0.121)	(0.156)	(0.165)	(0.177)	(0.198)	(0.244)
16	-0.312	-0.077	-0.278	0.016	0.079	0.332	0.359
	(0.237)	(0.217)	(0.241)	(0.306)	(0.405)	(0.349)	(0.484)

Notes: This table reports $\hat{\beta}_k$, for $k \in \{2, 1, 0, -1, -2, -3, -4\}$ from the estimation of equation (2) for 16 industries and overall using yearly data from 1990 to 2016, but using log earnings per worker instead of log establishment counts as the dependent variable. All regressions include the log of working-age population as control, as well as commuting zone–state fixed effects and pair–year fixed effects. Industries are sorted according to the 1990 earnings ranking of Table 1. Standard errors (in parentheses) are two-way clustered at the state and border segment levels. The coefficients are statistically significant at the *10%, **5%, or ***1% level.

B Theoretical Appendix

B.1 Household Maximization Problem

Given the utility function in (3), the representative household maximizes its utility by choosing N and allocating labor, $l(\omega)$, across firms. With the final good being the numéraire, consumption of the representative household, C, equals the wage income. Therefore, we can write the household's problem as

$$\max_{l(\omega)} \int_{\omega\in\Omega} w(\omega)l(\omega) - \frac{N^{1+\frac{1}{\psi}}}{1+\frac{1}{\psi}} \quad \text{subject to} \quad N = \left(\int_{\omega\in\Omega} l(\omega)^{\frac{1+\theta}{\theta}} d\omega\right)^{\frac{\theta}{1+\theta}}.$$

The solution to this problem yields that the labor supply to firm ω is

$$l(\omega) = N^{\frac{\psi-\theta}{\psi}} w(\omega)^{\theta}. \tag{B-1}$$

Using the definition of the wage index, $W \equiv \left(\int w(\omega)^{1+\theta}\right)^{\frac{1}{1+\theta}}$, it follows that

$$N^{\frac{1}{\theta}} \equiv \left(\int_{\omega\in\Omega} l(\omega)^{\frac{1+\theta}{\theta}}\right)^{\frac{1}{1+\theta}} = N^{\frac{\psi-\theta}{\psi\theta}} \left(\int_{\omega\in\Omega} w(\omega)^{1+\theta}\right)^{\frac{1}{1+\theta}} = N^{\frac{\psi-\theta}{\psi\theta}} W,$$

and hence $N = W^{\psi}$, so that we can rewrite the firm-level labor supply in (B-1) as

$$l(\omega) = \frac{w(\omega)^{\theta}}{W^{\theta-\psi}}. \tag{B-2}$$

Using (B-2), the aggregate labor supply is $L = \int_{\omega\in\Omega} l(\omega)d\omega = \frac{1}{W^{\theta-\psi}} \int_{\omega\in\Omega} w(\omega)^{\theta} d\omega$. Lastly, the aggregate wage bill is given by

$$\int_{\omega\in\Omega} w(\omega)l(\omega)d\omega = N^{\frac{\psi-\theta}{\psi}} \int_{\omega\in\Omega} w(\omega)^{1+\theta} d\omega = N^{\frac{\psi-\theta}{\psi}} W^{1+\theta} = N^{1-\frac{\theta}{\psi}} W^{1+\theta} = NW = W^{1+\psi}, \tag{B-3}$$

where first equality follows from (B-1), the second equality follows from the definition of W, and the third and fourth equalities follow from $N = W^{\psi}$.

B.2 Decentralized Equilibrium

In the free-entry condition in (10), the left-hand side is strictly decreasing with $\hat{\varphi}$, approaching zero as $\hat{\varphi} \to \infty$. Thus, as long as f_E is sufficiently small, there is unique value for $\hat{\varphi}$ that solves (10). From the zero-cutoff-profit condition, the solution for the wage index is then

$$W_D = \left\{\left[\frac{\theta^{\theta}}{(1+\theta)^{1+\theta}}\right] \frac{\hat{\varphi}_D^{1+\theta}}{f}\right\}^{\frac{1}{\theta-\psi}}, \tag{B-4}$$

and from (8) and (B-4) we know that

$$l_D(\varphi) = \left(\frac{\theta}{1+\theta}\right)^{\theta} \frac{\varphi^{\theta}}{W_D^{\theta-\psi}} = \frac{(1+\theta)f\varphi^{\theta}}{\hat{\varphi}_D^{1+\theta}}. \tag{B-5}$$

To obtain the expression for the equilibrium mass of firms, M_D, we use equations (11) and (8) to obtain

$$M_D = \left[\int_{\hat{\varphi}_D}^{\infty} \varphi^{1+\theta} g(\varphi|\varphi \geq \hat{\varphi}_D) d\varphi\right]^{-1} \left(\frac{1+\theta}{\theta}\right)^{1+\theta} W_D^{1+\theta} \tag{B-6}$$

From equation (12), the equilibrium total employment is

$$L_D = \frac{(1+\theta) f M_D}{\hat{\varphi}_D^{1+\theta}} \int_{\hat{\varphi}_D}^{\infty} \varphi^{\theta} g(\varphi|\varphi \geq \hat{\varphi}) d\varphi. \tag{B-7}$$

Finally, from (7) we know that $\mathbb{U}_D = \left(\frac{1}{1+\psi}\right) W_D^{1+\psi} = \left(\frac{1}{1+\psi}\right) N_D^{1+\frac{1}{\psi}}$, where the second equality follows from $N_D = W_D^{\psi}$.

B.3 Social Planner's Problem

The social planner chooses the cutoff productivity level ($\hat{\varphi}$), the mass of entrants (M_E), and firm-level labor supplies ($l(\varphi)$ for every $\varphi \geq \hat{\varphi}$) that maximize the household utility function in (3) subject to equation (4) and $C = M \int_{\hat{\varphi}}^{\infty} \varphi l(\varphi) g(\varphi|\varphi \geq \hat{\varphi}) d\varphi - M_E f_E - Mf$, for $M = [1 - G(\hat{\varphi})] M_E$. Notice that household consumption equals total output minus entry costs and fixed costs of operation (recall that f and f_E are in terms of the final good). Using $g(\varphi|\varphi \geq \hat{\varphi}) = \frac{g(\varphi)}{1-G(\hat{\varphi})}$, we can write the planner's problem as

$$\max_{l(\varphi), M_E, \hat{\varphi}} \left\{ M_E \int_{\hat{\varphi}}^{\infty} \varphi l(\varphi) g(\varphi) d\varphi - M_E f_E - [1 - G(\hat{\varphi})] M_E f - \frac{\overbrace{\left[\left(M_E \int_{\hat{\varphi}}^{\infty} l(\varphi)^{\frac{1+\theta}{\theta}} g(\varphi) d\varphi\right)^{\frac{\theta}{1+\theta}}\right]}^{N}{}^{1+\frac{1}{\psi}}}{1 + 1/\psi} \right\}. \tag{B-8}$$

The first order conditions with respect to $l(\varphi)$, M_E, and $\hat{\varphi}$ are respectively

$$\varphi - N^{\frac{\theta-\psi}{\theta\psi}} l(\varphi)^{\frac{1}{\theta}} = 0, \tag{B-9}$$

$$\int_{\hat{\varphi}}^{\infty} \varphi l(\varphi) g(\varphi) d\varphi - \left(\frac{\theta}{1+\theta}\right) N^{\frac{\theta-\psi}{\theta\psi}} \int_{\hat{\varphi}}^{\infty} l(\varphi)^{\frac{1+\theta}{\theta}} g(\varphi) d\varphi - f_E - [1 - G(\hat{\varphi})] f = 0, \tag{B-10}$$

$$\hat{\varphi} l(\hat{\varphi}) - \left(\frac{\theta}{1+\theta}\right) N^{\frac{\theta-\psi}{\theta\psi}} l(\hat{\varphi})^{\frac{1+\theta}{\theta}} - f = 0. \tag{B-11}$$

Note that (B-9) implies that $N^{\frac{\theta-\psi}{\theta\psi}} = \frac{\hat{\varphi}}{l(\hat{\varphi})^{\frac{1}{\theta}}}$, which plugged into (B-11) yields $l(\hat{\varphi}) = \frac{(1+\theta)f}{\hat{\varphi}}$, so that we can solve for N as a function of $\hat{\varphi}$ as

$$N = \left[\frac{\hat{\varphi}^{1+\theta}}{(1+\theta)f}\right]^{\frac{\psi}{\theta-\psi}}. \tag{B-12}$$

Note also from (B-9) that $\frac{l(\varphi)}{l(\hat{\varphi})} = \left(\frac{\varphi}{\hat{\varphi}}\right)^{\theta}$, which from (8) we know that it is the same allocation of resources across firms as in the decentralized case. This proves that the allocation of resources across firms in the decentralized case is efficient. Plugging in (B-12) into (B-9), we can write the planner's optimal firm-level labor supply as

$$l(\varphi) = \frac{(1+\theta) f \varphi^{\theta}}{\hat{\varphi}^{1+\theta}}. \tag{B-13}$$

Using (B-13), we obtain the following expressions

$$\int_{\hat{\varphi}}^{\infty} \varphi l\left(\varphi\right) g(\varphi) d\varphi = (1+\theta) f \int_{\hat{\varphi}}^{\infty} \left(\frac{\varphi}{\hat{\varphi}}\right)^{1+\theta} g(\varphi) d\varphi, \tag{B-14}$$

$$\int_{\hat{\varphi}}^{\infty} l\left(\varphi\right)^{\frac{1+\theta}{\theta}} g(\varphi) d\varphi = \left[\frac{(1+\theta) f}{\hat{\varphi}}\right]^{\frac{1+\theta}{\theta}} \int_{\hat{\varphi}}^{\infty} \left(\frac{\varphi}{\hat{\varphi}}\right)^{1+\theta} g(\varphi) d\varphi, \tag{B-15}$$

which along with (B-12) can be plugged into (B-10) to obtain

$$\int_{\hat{\varphi}}^{\infty} \left[\left(\frac{\varphi}{\hat{\varphi}}\right)^{1+\theta} - 1\right] f g(\varphi) d\varphi = f_E. \tag{B-16}$$

Notice that (B-16) is exactly the same as (10), which is the equation determining $\hat{\varphi}_D$ in the decentralized case. Using $\hat{\varphi}_P$ to denote the equilibrium cutoff productivity level in the planner's problem, it follows that $\hat{\varphi}_P = \hat{\varphi}_D$. Once we obtain $\hat{\varphi}_P$, we plug it into (B-12) and (B-13) to obtain N_P and $l_P(\varphi)$. From (B-5), it follows that $l_P(\varphi) = l_D(\varphi)$, so that firm size is the same in both cases. Moreover, using $N_D = W_D^{\psi}$, (B-4), (B-12), and $\hat{\varphi}_P = \hat{\varphi}_D$ we get that

$$\frac{N_P}{N_D} = \left(\frac{1+\theta}{\theta}\right)^{\frac{\theta\psi}{\theta-\psi}} > 1.$$

To obtain the mass of firms, we use the definition of N as written in the planner's maximization problem in (B-8), along with (B-13) and $M = \left[1 - G(\hat{\varphi})\right] M_E$ to get

$$M_P = \left\{\left[\frac{(1+\theta) f}{\hat{\varphi}_P^{1+\theta}}\right]^{\frac{1+\theta}{\theta}} \int_{\hat{\varphi}_P}^{\infty} \varphi^{1+\theta} g(\varphi|\varphi \geq \hat{\varphi}_P) d\varphi\right\}^{-1} N_P^{\frac{1+\theta}{\theta}} = \left[\int_{\hat{\varphi}_P}^{\infty} \varphi^{1+\theta} g(\varphi|\varphi \geq \hat{\varphi}_P) d\varphi\right]^{-1} N_P^{\frac{1+\theta}{\psi}}, \tag{B-17}$$

where the second equality follows from (B-12). From (B-17) and (B-6), note that the ratio between M_P and M_D is $\frac{M_P}{M_D} = \left[\frac{\theta N_P^{1/\psi}}{(1+\theta) W_D}\right]^{1+\theta}$, which using (B-4), (B-12), and $\hat{\varphi}_P = \hat{\varphi}_D$, can be rewritten as

$$\frac{M_P}{M_D} = \left(\frac{1+\theta}{\theta}\right)^{\frac{(1+\theta)\psi}{\theta-\psi}} > 1. \tag{B-18}$$

Thus, the decentralized outcome yields a suboptimal mass of firms. Total employment in the planner's case is $L_P = M_P \int_{\hat{\varphi}_P}^{\infty} l_P(\varphi) g(\varphi|\varphi \geq \hat{\varphi}_P) d\varphi$, which using (B-13) can be rewritten as

$$L_P = \frac{(1+\theta) f M_P}{\hat{\varphi}_P^{1+\theta}} \int_{\hat{\varphi}_D}^{\infty} \varphi^{\theta} g(\varphi|\varphi \geq \hat{\varphi}_D) d\varphi. \tag{B-19}$$

From (B-19), (B-7), and $\hat{\varphi}_P = \hat{\varphi}_D$, it follows that

$$\frac{L_P}{L_D} = \frac{M_P}{M_D} = \left(\frac{1+\theta}{\theta}\right)^{\frac{(1+\theta)\psi}{\theta-\psi}} > 1. \tag{B-20}$$

Regarding welfare, note first from (B-16) that $M_E f_E + [1-G(\hat{\varphi})]M_E f = M_E \int_{\hat{\varphi}}^{\infty} \left(\frac{\varphi}{\hat{\varphi}}\right)^{1+\theta} f g(\varphi) d\varphi$, which along with (B-14) and $M_E = \frac{M}{1-G(\hat{\varphi})}$ can be plugged into the maximized value of welfare in (B-8) to get

$$\mathbb{U}_P = \frac{M_P \theta f}{\hat{\varphi}_P^{1+\theta}} \int_{\hat{\varphi}_P}^{\infty} \varphi^{1+\theta} g(\varphi|\varphi \geq \hat{\varphi}_P) d\varphi - \frac{N_P^{1+\frac{1}{\psi}}}{1+1/\psi} = \left(\frac{\theta f}{\hat{\varphi}_P^{1+\theta}}\right) N_P^{\frac{1+\theta}{\psi}} - \frac{N_P^{1+\frac{1}{\psi}}}{1+1/\psi}, \tag{B-21}$$

where the second equality uses (B-17). From (B-12), we obtain that $\frac{\theta f}{\hat{\varphi}_P^{1+\theta}} = \frac{\theta}{(1+\theta) N_P^{(\theta-\psi)/\psi}}$, which plugged into (B-21) yields that welfare in the planner's case is

$$\mathbb{U}_P = \frac{(\theta-\psi) N_P^{1+\frac{1}{\psi}}}{(1+\theta)(1+\psi)}. \tag{B-22}$$

Using the expression for welfare in the decentralized case given in the end of section B.2, we get that $\frac{\mathbb{U}_P}{\mathbb{U}_D} = \frac{\theta-\psi}{1+\theta}\left(\frac{N_P}{N_D}\right)^{1+1/\psi}$. Using (B-12), $N_D = W_D^{\psi}$, (B-4), and $\hat{\varphi}_P = \hat{\varphi}_D$, this ratio can be rewritten as

$$\frac{\mathbb{U}_P}{\mathbb{U}_D} = \left(\frac{\theta-\psi}{1+\theta}\right)\left(\frac{1+\theta}{\theta}\right)^{\frac{\theta(1+\psi)}{\theta-\psi}} > 1. \tag{B-23}$$

We know that $\frac{\mathbb{U}_P}{\mathbb{U}_D} > 1$ because $\frac{\mathbb{U}_P}{\mathbb{U}_D}$ is strictly decreasing in θ,

$$\frac{d\left(\mathbb{U}_P/\mathbb{U}_D\right)}{d\theta} = -\frac{\psi(1+\psi)}{(\theta-\psi)(1+\theta)}\left(\frac{1+\theta}{\theta}\right)^{\frac{\theta(1+\psi)}{\theta-\psi}} \ln\left(\frac{1+\theta}{\theta}\right) < 0,$$

and $\lim_{\theta\to\infty} \frac{\mathbb{U}_P}{\mathbb{U}_D} = 1$.

B.4 Optimality of Labor Subsidy/Leisure Tax

Suppose the government subsidizes labor at rate s and finances it using a lump sum tax on workers, T. The utility maximization exercise is then

$$\max_{l(\omega)} \int_{\omega\in\Omega} (1+s) w(\omega) l(\omega) - T - \frac{N^{1+\frac{1}{\psi}}}{1+\frac{1}{\psi}} \quad \text{subject to} \quad N = \left(\int_{\omega\in\Omega} l(\omega)^{\frac{1+\theta}{\theta}} d\omega\right)^{\frac{\theta}{1+\theta}},$$

which yields the following labor supply curve to firm ω:

$$l_S(\omega) = N_S^{\frac{\psi-\theta}{\psi}} (1+s)^{\theta} w(\omega)^{\theta}. \tag{B-24}$$

We use subscript S to indicate that this is the model with a wage subsidy.

Using the definition of the wage index, W, it follows that

$$N_S^{\frac{1}{\theta}} \equiv \left(\int_{\omega\in\Omega} l(\omega)^{\frac{1+\theta}{\theta}}\right)^{\frac{1}{1+\theta}} = N_S^{\frac{\psi-\theta}{\psi\theta}}(1+s)\left(\int_{\omega\in\Omega} w(\omega)^{1+\theta}\right)^{\frac{1}{1+\theta}} = N_S^{\frac{\psi-\theta}{\psi\theta}}(1+s)W_S,$$

and hence $N_S = (1+s)^{\psi}W_S^{\psi}$, so that we can rewrite the firm-level labor supply in (B-24) as

$$l_S(\omega) = (1+s)^{\psi}W_S^{\psi-\theta}w(\omega)^{\theta} = \frac{(1+s)^{\psi}w(\omega)^{\theta}}{W_S^{\theta-\psi}}. \tag{B-25}$$

We can also verify that the total wage bill is still given by $W_S N_S$:

$$\int_{\omega\in\Omega} w(\omega)l(\omega)d\omega = N_S^{\frac{\psi-\theta}{\psi}}(1+s)^{\theta}\int_{\omega\in\Omega} w(\omega)^{1+\theta}d\omega = (1+s)^{\theta}N_S^{1-\frac{\theta}{\psi}}W_S^{1+\theta} = W_S N_S.$$

From the firm optimization problem we obtain

$$w_S(\varphi) = \left(\frac{\theta}{1+\theta}\right)\varphi \quad \text{and} \quad l_S(\varphi) = \left(\frac{\theta}{1+\theta}\right)^{\theta}\frac{\varphi^{\theta}}{W_S^{\theta-\psi}}(1+s)^{\psi}, \tag{B-26}$$

and hence the gross profit is

$$\pi_S(\varphi) = \left(\frac{\theta^{\theta}}{(1+\theta)^{1+\theta}}\right)\frac{(1+s)^{\psi}\varphi^{1+\theta}}{W_S^{\theta-\psi}}.$$

The zero cutoff profit condition is $\pi_S(\hat{\varphi}_S) = f$. Since the free entry condition given in (10) is unchanged, $\hat{\varphi}$ remains unchanged: $\hat{\varphi}_S = \hat{\varphi}_D = \hat{\varphi}_P$. Therefore, W_S is determined by the zero cutoff profit condition and is given by

$$\left(\frac{\theta^{\theta}}{(1+\theta)^{1+\theta}}\right)\frac{(1+s)^{\psi}\hat{\varphi}_S^{1+\theta}}{W_S^{\theta-\psi}} = f, \tag{B-27}$$

which implies that

$$W_S = \left\{\left(\frac{\theta^{\theta}}{(1+\theta)^{1+\theta}}\right)\frac{\hat{\varphi}^{1+\theta}(1+s)^{\psi}}{f}\right\}^{\frac{1}{\theta-\psi}}, \tag{B-28}$$

From $N_S = (1+s)^{\psi}W_S^{\psi}$, it follows that

$$N_S = (1+s)^{\frac{\theta\psi}{\theta-\psi}}\left\{\left(\frac{\theta^{\theta}}{(1+\theta)^{1+\theta}}\right)\frac{\hat{\varphi}^{1+\theta}}{f}\right\}^{\frac{\psi}{\theta-\psi}}. \tag{B-29}$$

Recall that in the planner's problem $N_P = \left[\frac{\hat{\varphi}^{1+\theta}}{(1+\theta)f}\right]^{\frac{\psi}{\theta-\psi}}$. Therefore, $N_S = N_P$ if and only if $s = \frac{1}{\theta}$. Thus, a labor subsidy or a leisure tax in the amount of $s = \frac{1}{\theta}$ yields the same N as in the planner's problem.

Next, note that using (B-28), the expression for firm-level employment given in (B-26) becomes

$$l_S(\varphi) = \frac{(1+\theta)f\varphi^{\theta}}{\hat{\varphi}_S^{1+\theta}}. \tag{B-30}$$

That is, this expression remains the same as in the decentralized equilibrium without policy intervention and in the planner's problem. From the definition of N_S,

$$N_S \equiv \left[M_S \int_{\hat{\varphi}_S}^{\infty} l\left(\varphi\right)^{\frac{1+\theta}{\theta}} g(\varphi|\varphi \geq \hat{\varphi}_S) d\varphi \right]^{\frac{\theta}{1+\theta}},$$

it follows that $M_S = M_P$ as given by (B-17).

We have seen that the total wage bill is still given by $W_S N_S$. Therefore, the consumption of households is $(1+s)W_S N_S - T = W_S N_S$ because of balanced budget. We have shown that $N_S = (1+s)^{\psi} W_S^{\psi}$, and hence $W_S N_S = \frac{N_S^{1+\frac{1}{\psi}}}{1+s} = \left(\frac{\theta}{1+\theta}\right) N_S^{1+\frac{1}{\psi}}$. This exactly equals the expression for consumption in the planner's problem because $N_S = N_P$. Hence, $\mathbb{U}_S = \mathbb{U}_P$. Thus, a proportional labor subsidy (or leisure tax) of $s = \frac{1}{\theta}$ restores optimality.

B.5 The Effects of a Binding Minimum Wage

This section presents the proof of Proposition 1. For a binding minimum wage $\underline{w}$, so that $\underline{w} > w(\hat{\varphi}_D)$, we show that for every productivity distribution, $\frac{d\hat{\underline{\varphi}}}{d\underline{w}} > 0$, $\frac{dM}{d\underline{w}} < 0$, $\frac{d\underline{W}}{d\underline{w}} < 0$, $\frac{d\mathbb{U}}{d\underline{w}} < 0$, and that if the productivity distribution is Pareto, then it also holds that $\frac{dL}{d\underline{w}} < 0$ and $\frac{d\overline{w}}{d\underline{w}} > 0$.

From the zero-cutoff-profit condition in section 3.3, we know that $\underline{W}^{\theta-\psi} = (\hat{\underline{\varphi}} - \underline{w})\frac{\underline{w}^{\theta}}{f}$, which allows us to rewrite the free-entry condition in (13) as

$$\int_{\hat{\underline{\varphi}}}^{\underline{\varphi}} \left(\frac{\varphi - \underline{w}}{\hat{\underline{\varphi}} - \underline{w}} - 1 \right) f g(\varphi) d\varphi + \int_{\underline{\varphi}}^{\infty} \left\{ \left[\frac{\theta^{\theta}}{(1+\theta)^{1+\theta}} \right] \frac{\varphi^{1+\theta}}{(\hat{\underline{\varphi}} - \underline{w})\underline{w}^{\theta}} - 1 \right\} f g(\varphi) d\varphi = f_E. \tag{B-31}$$

Taking the derivative of (B-31) with respect to $\underline{w}$, and given that $\frac{d\underline{\varphi}}{d\underline{w}} = \frac{1+\theta}{\theta}$, we obtain that

$$\frac{d\hat{\underline{\varphi}}}{d\underline{w}} = \left[\frac{\int_{\hat{\underline{\varphi}}}^{\underline{\varphi}} \left(\frac{\varphi - \hat{\underline{\varphi}}}{\underline{\varphi} - \hat{\underline{\varphi}}}\right) g(\varphi) d\varphi + \int_{\underline{\varphi}}^{\infty} \left(\frac{\varphi}{\underline{\varphi}}\right)^{1+\theta} g(\varphi) d\varphi}{\int_{\hat{\underline{\varphi}}}^{\underline{\varphi}} \left(\frac{\varphi - \underline{w}}{\underline{\varphi} - \underline{w}}\right) g(\varphi) d\varphi + \int_{\underline{\varphi}}^{\infty} \left(\frac{\varphi}{\underline{\varphi}}\right)^{1+\theta} g(\varphi) d\varphi} \right] \left(\frac{\underline{\varphi} - \hat{\underline{\varphi}}}{\underline{\varphi} - \underline{w}} \right) > 0. \tag{B-32}$$

All the terms in (B-32) are positive because $\underline{w} < \hat{\underline{\varphi}} < \underline{\varphi}$. We know that $\underline{w} < \hat{\underline{\varphi}}$ from the zero-cutoff-profit condition, $(\hat{\underline{\varphi}} - \underline{w})\frac{\underline{w}^{\theta}}{\underline{W}^{\theta-\psi}} = f$, and we know that $\hat{\underline{\varphi}} < \underline{\varphi}$ because firms with $\varphi \in [\hat{\underline{\varphi}}, \underline{\varphi})$ are constrained by the minimum wage. Moreover, the term in brackets is less than 1 because $\int_{\hat{\underline{\varphi}}}^{\underline{\varphi}} \left(\frac{\varphi - \hat{\underline{\varphi}}}{\underline{\varphi} - \hat{\underline{\varphi}}}\right) g(\varphi) d\varphi < \int_{\hat{\underline{\varphi}}}^{\underline{\varphi}} \left(\frac{\varphi - \underline{w}}{\underline{\varphi} - \underline{w}}\right) g(\varphi) d\varphi$, which follows from $\frac{(\hat{\underline{\varphi}} - \underline{w})(\underline{\varphi} - \varphi)}{(\underline{\varphi} - \underline{w})(\underline{\varphi} - \hat{\underline{\varphi}})} > 0$ for every $\varphi < \underline{\varphi}$. Therefore,

$$\frac{d\hat{\underline{\varphi}}}{d\underline{w}} \in \left(0, \frac{\underline{\varphi} - \hat{\underline{\varphi}}}{\underline{\varphi} - \underline{w}} \right). \tag{B-33}$$

From the zero-cutoff-profit condition we get that $\underline{W} = \left[\frac{(\hat{\underline{\varphi}} - \underline{w})\underline{w}^{\theta}}{f} \right]^{1/(\theta - \psi)}$, and thus

$$\frac{d\underline{W}}{d\underline{w}} = \frac{\underline{w}^{\theta}}{(\theta - \psi)\underline{W}^{\theta-\psi-1} f} \left[\frac{\theta(\hat{\underline{\varphi}} - \underline{w})}{\underline{w}} + \frac{d\hat{\underline{\varphi}}}{d\underline{w}} - 1 \right].$$

The sign of $\frac{dW}{d\underline{w}}$ is determined by the term within the brackets, which is negative if and only if

$$\frac{d\underline{\hat{\varphi}}}{d\underline{w}} < 1 - \frac{\theta(\underline{\hat{\varphi}} - \underline{w})}{\underline{w}} = 1 - \frac{\underline{\hat{\varphi}} - \underline{w}}{\underline{\varphi} - \underline{w}} = \frac{\underline{\varphi} - \underline{\hat{\varphi}}}{\underline{\varphi} - \underline{w}}, \tag{B-34}$$

where we use that $\underline{\varphi} - \underline{w} = \frac{\underline{w}}{\theta}$. From (B-33) we know that (B-34) holds, and therefore, $\frac{d\underline{W}}{d\underline{w}} < 0$. Similar to (7), welfare with a binding minimum wage is given by $\underline{\mathbb{U}} = \frac{\underline{W}^{1+\psi}}{1+\psi}$, and thus, $\frac{d\underline{\mathbb{U}}}{d\underline{w}} = \underline{W}^{\psi}\frac{d\underline{W}}{d\underline{w}} < 0$.

The wage index is defined as $\underline{W} = \left[\underline{M}\int_{\underline{\hat{\varphi}}}^{\infty} w(\varphi)^{1+\theta} g(\varphi|\varphi \geq \underline{\hat{\varphi}})d\varphi\right]^{1/(1+\theta)}$ where $w(\varphi) = \underline{w}$ for $\varphi \in [\underline{\hat{\varphi}}, \underline{\varphi})$ and $w(\varphi) = \left(\frac{\theta}{1+\theta}\right)\varphi$ for $\varphi \geq \underline{\varphi}$. Thus, we can solve for the mass of firms as

$$\underline{M} = \frac{[1 - G(\underline{\hat{\varphi}})]\underline{W}^{1+\theta}}{\underline{w}^{1+\theta}[G(\underline{\varphi}) - G(\underline{\hat{\varphi}})] + \int_{\underline{\varphi}}^{\infty} w(\varphi)^{1+\theta} g(\varphi)d\varphi}.$$

Taking the derivative of $\ln \underline{M}$ with respect to $\underline{w}$ yields

$$\begin{aligned}
\frac{d\ln \underline{M}}{d\underline{w}} =& (1+\theta)\frac{d\ln \underline{W}}{d\underline{w}} - \left[\frac{g(\underline{\hat{\varphi}})}{1 - G(\underline{\hat{\varphi}})}\right]\frac{d\underline{\hat{\varphi}}}{d\underline{w}} + \frac{\underline{w}^{1+\theta} g(\underline{\hat{\varphi}})\frac{d\underline{\hat{\varphi}}}{d\underline{w}} - (1+\theta)\underline{w}^{\theta}[G(\underline{\varphi}) - G(\underline{\hat{\varphi}})]}{\underline{w}^{1+\theta}[G(\underline{\varphi}) - G(\underline{\hat{\varphi}})] + \int_{\underline{\varphi}}^{\infty} w(\varphi)^{1+\theta} g(\varphi)d\varphi} \\
=& \underbrace{(1+\theta)\frac{d\ln \underline{W}}{d\underline{w}}}_{<0} - g(\underline{\hat{\varphi}})\frac{d\underline{\hat{\varphi}}}{d\underline{w}}\left[\frac{1}{1 - G(\underline{\hat{\varphi}})} - \frac{1}{G(\underline{\varphi}) - G(\underline{\hat{\varphi}}) + \int_{\underline{\varphi}}^{\infty} [w(\varphi)/\underline{w}]^{1+\theta} g(\varphi)d\varphi}\right] \\
& \underbrace{- \left(\frac{1+\theta}{\underline{w}}\right)\frac{G(\underline{\varphi}) - G(\underline{\hat{\varphi}})}{G(\underline{\varphi}) - G(\underline{\hat{\varphi}}) + \int_{\underline{\varphi}}^{\infty} [w(\varphi)/\underline{w}]^{1+\theta} g(\varphi)d\varphi}}_{<0}.
\end{aligned} \tag{B-35}$$

Thus, a sufficient condition for $\frac{d\ln \underline{M}}{d\underline{w}} < 0$ is that

$$-g(\underline{\hat{\varphi}})\frac{d\underline{\hat{\varphi}}}{d\underline{w}}\left[\frac{1}{1 - G(\underline{\hat{\varphi}})} - \frac{1}{G(\underline{\varphi}) - G(\underline{\hat{\varphi}}) + \int_{\underline{\varphi}}^{\infty} [w(\varphi)/\underline{w}]^{1+\theta} g(\varphi)d\varphi}\right] < 0,$$

which is true if $\int_{\underline{\varphi}}^{\infty}\left[\frac{w(\varphi)}{\underline{w}}\right]^{1+\theta} g(\varphi)d\varphi > 1 - G(\underline{\varphi})$, which implies $\int_{\underline{\varphi}}^{\infty}\left[\frac{w(\varphi)}{\underline{w}}\right]^{1+\theta} g(\varphi|\varphi \geq \underline{\varphi})d\varphi > 1$. This condition holds because $\frac{w(\varphi)}{\underline{w}} \geq 1$ for $\varphi \geq \underline{\varphi}$ (with equality if and only if $\varphi = \underline{\varphi}$). Therefore, $\frac{d\underline{M}}{d\underline{w}} < 0$.

Total employment is defined as $\underline{L} = \underline{M}\int_{\underline{\hat{\varphi}}}^{\infty} l(\varphi)g(\varphi|\varphi \geq \underline{\hat{\varphi}})d\varphi$ where $l(\varphi) = \frac{\underline{w}^{\theta}}{\underline{W}^{\theta-\psi}}$ for $\varphi \in [\underline{\hat{\varphi}}, \underline{\varphi})$ and $l(\varphi) = \frac{w(\varphi)^{\theta}}{\underline{W}^{\theta-\psi}}$ for $\varphi \geq \underline{\varphi}$. Hence, we can rewrite $\underline{L}$ as

$$\underline{L} = \frac{\underline{M}\underline{w}^{\theta}}{[1 - G(\underline{\hat{\varphi}})]\underline{W}^{\theta-\psi}}\left\{G(\underline{\varphi}) - G(\underline{\hat{\varphi}}) + \int_{\underline{\varphi}}^{\infty}\left[\frac{w(\varphi)}{\underline{w}}\right]^{\theta} g(\varphi)d\varphi\right\}. \tag{B-36}$$

The derivative of $\ln \underline{L}$ with respect to $\underline{w}$ is then given by

$$
\begin{aligned}
\frac{d\ln \underline{L}}{d\underline{w}} =& \frac{d\ln \underline{M}}{d\underline{w}} - (\theta-\psi)\frac{d\ln \underline{W}}{d\underline{w}} + g(\underline{\hat{\varphi}})\frac{d\underline{\hat{\varphi}}}{d\underline{w}}\left[\frac{1}{1-G(\underline{\hat{\varphi}})} - \frac{1}{G(\underline{\varphi})-G(\underline{\hat{\varphi}})+\int_{\underline{\varphi}}^{\infty}\left[{w(\varphi)}/{\underline{w}}\right]^{\theta} g(\varphi)d\varphi}\right] \\
&+\left(\frac{\theta}{\underline{w}}\right)\frac{G(\underline{\varphi})-G(\underline{\hat{\varphi}})}{G(\underline{\varphi})-G(\underline{\hat{\varphi}})+\int_{\underline{\varphi}}^{\infty}\left[{w(\varphi)}/{\underline{w}}\right]^{\theta} g(\varphi)d\varphi} \\
=& -g(\underline{\hat{\varphi}})\frac{d\underline{\hat{\varphi}}}{d\underline{w}}\left[\frac{1}{G(\underline{\varphi})-G(\underline{\hat{\varphi}})+\int_{\underline{\varphi}}^{\infty}\left[{w(\varphi)}/{\underline{w}}\right]^{\theta} g(\varphi)d\varphi} - \frac{1}{G(\underline{\varphi})-G(\underline{\hat{\varphi}})+\int_{\underline{\varphi}}^{\infty}\left[{w(\varphi)}/{\underline{w}}\right]^{1+\theta} g(\varphi)d\varphi}\right] \\
&+(1+\psi)\frac{d\ln \underline{W}}{d\underline{w}} + \frac{G(\underline{\varphi})-G(\underline{\hat{\varphi}})}{\underline{w}}\times \\
&\left[\frac{\theta}{G(\underline{\varphi})-G(\underline{\hat{\varphi}})+\int_{\underline{\varphi}}^{\infty}\left[{w(\varphi)}/{\underline{w}}\right]^{\theta} g(\varphi)d\varphi} - \frac{1+\theta}{G(\underline{\varphi})-G(\underline{\hat{\varphi}})+\int_{\underline{\varphi}}^{\infty}\left[{w(\varphi)}/{\underline{w}}\right]^{1+\theta} g(\varphi)d\varphi}\right] \\
=& -\frac{1}{D}\Bigg\{\underbrace{\left[g(\underline{\hat{\varphi}})\frac{d\underline{\hat{\varphi}}}{d\underline{w}} - \frac{\theta[G(\underline{\varphi})-G(\underline{\hat{\varphi}})]}{\underline{w}}\right]\left[\int_{\underline{\varphi}}^{\infty}\left[\left(\frac{w(\varphi)}{\underline{w}}\right)^{1+\theta} - \left(\frac{w(\varphi)}{\underline{w}}\right)^{\theta}\right] g(\varphi)d\varphi\right]}_{\text{Term 1}} \\
&+\underbrace{\left[\frac{G(\underline{\varphi})-G(\underline{\hat{\varphi}})}{\underline{w}}\right]\left[G(\underline{\varphi})-G(\underline{\hat{\varphi}})+\int_{\underline{\varphi}}^{\infty}\left[\frac{w(\varphi)}{\underline{w}}\right]^{\theta} g(\varphi)d\varphi\right]}_{\text{Term 2}}\Bigg\} \underbrace{+(1+\psi)\frac{d\ln \underline{W}}{d\underline{w}}}_{<0}, \qquad \text{(B-37)}
\end{aligned}
$$

where $D = \{G(\underline{\varphi})-G(\underline{\hat{\varphi}})+\int_{\underline{\varphi}}^{\infty}\left[{w(\varphi)}/{\underline{w}}\right]^{\theta} g(\varphi)d\varphi\}\{G(\underline{\varphi})-G(\underline{\hat{\varphi}})+\int_{\underline{\varphi}}^{\infty}\left[{w(\varphi)}/{\underline{w}}\right]^{1+\theta} g(\varphi)d\varphi\} > 0$. The second equality above uses (B-35) to substitute for $\frac{d\ln \underline{M}}{d\underline{w}}$. Without further assumptions on the productivity distribution, we cannot pin down the sign of Term 1 + Term 2 in (B-37). However, if we assume a Pareto distribution for productivity, $g(\varphi) = \frac{k}{\varphi^{k+1}}$ and $G(\varphi) = 1 - \frac{1}{\varphi^k}$ for $k > 1+\theta$, we can show that Term 1 + Term 2 > 0, so that $\frac{d\ln \underline{L}}{d\underline{w}} < 0$.

Using $\frac{w(\varphi)}{\underline{w}} = \frac{\varphi}{\underline{\varphi}}$ for $\varphi \geq \underline{\varphi}$, $g(\underline{\hat{\varphi}}) = \frac{k}{\underline{\hat{\varphi}}^{k+1}}$, $G(\underline{\varphi}) - G(\underline{\hat{\varphi}}) = \frac{1}{\underline{\hat{\varphi}}^k} - \frac{1}{\underline{\varphi}^k}$, $\int_{\underline{\varphi}}^{\infty}\left(\frac{\varphi}{\underline{\varphi}}\right)^{\theta} g(\varphi)d\varphi = \left(\frac{k}{k-\theta}\right)\frac{1}{\underline{\varphi}^k}$, $\int_{\underline{\varphi}}^{\infty}\left(\frac{\varphi}{\underline{\varphi}}\right)^{1+\theta} g(\varphi)d\varphi = \left(\frac{k}{k-\theta-1}\right)\frac{1}{\underline{\varphi}^k}$, and defining $u \equiv \frac{\underline{\varphi}}{\underline{\hat{\varphi}}} \in \left(1, \frac{1+\theta}{\theta}\right)$, we obtain that Term 1+Term 2 $>$ 0 if and only if

$$
\underbrace{\frac{u^k}{u^k-1}\left\{\frac{\frac{u^k}{k-1} + \frac{k\theta u}{(k-1)(k-\theta-1)} - \frac{1+\theta}{k-\theta-1}}{\frac{(1+\theta)u^{k-1}}{k-1} - \frac{\theta u^k}{k} + \frac{\theta(1+\theta)}{k(k-1)(k-\theta-1)}}\right\}}_{\text{Term 3}} + \underbrace{\left(\frac{k-\theta-1}{\theta}\right)\left[1+\left(\frac{k-\theta}{k}\right)(u^k-1)\right]}_{>0} > 1. \qquad \text{(B-38)}
$$

Thus, a sufficient condition for (B-38) to hold is that Term 3 > 1. In the term in braces within Term 3, both the numerator and denominator are positive and increasing in u, with the numerator approaching 0 and the denominator approaching $\frac{1}{k-\theta-1}$ as $u \to 1$. Rearranging terms, we get that

Term 3 > 1 if and only if

$$\frac{\theta(1+\theta)}{(k-\theta-1)u^k}+\frac{(1+\theta)k}{u}-(k-1)\theta > (1+\theta)ku^{k-1}-[(k-1)\theta+k]u^k-\frac{k^2\theta u}{k-\theta-1}+\frac{(1+\theta)[k(k-1)+\theta]}{k-\theta-1}. \tag{B-39}$$

Both the left-hand side (LHS) and the right-hand side (RHS) approach $\frac{k(k-1)}{k-\theta-1}$ as $u \to 1$, and they are both decreasing in u, with

$$\begin{aligned}\frac{d\text{LHS}}{du} &= -\,k(1+\theta)\left[\frac{1}{u^2}+\frac{\theta}{(k-\theta-1)u^{k+1}}\right] < 0,\\ \frac{d\text{RHS}}{du} &= -\,k\left[[k(1+\theta)-\theta]u^{k-1}\left(1-\frac{1}{u}\right)+u^{k-2}+\frac{k\theta}{k-\theta-1}\right] < 0.\end{aligned}$$

Hence, if the LHS declines faster with u than the RHS, then it must be that condition (B-39) holds. We obtain that $\left|\frac{d\text{LHS}}{du}\right| > \left|\frac{d\text{RHS}}{du}\right|$ if

$$\underbrace{u^k\left[\left(k-\frac{\theta}{1+\theta}\right)u-k+1\right]+\frac{k\theta u^2}{(1+\theta)(k-\theta-1)}}_{\text{Term 4}} > \underbrace{1+\frac{\theta}{(k-\theta-1)u^{k-1}}}_{\text{Term 5}},$$

which is always true because both Term 4 and Term 5 approach $\frac{k-1}{k-\theta-1}$ as $u \to 1$, and Term 4 is strictly increasing with u, whereas Term 5 is strictly decreasing with u. Therefore, Term 3 > 1, and thus, Term 1+Term 2 > 0 and $\frac{d\ln \underline{L}}{d\underline{w}} < 0$.

From (B-3) we know that the wage bill is equal to $\underline{W}^{1+\psi}$. It is also the case that the wage bill equals the product of the average wage and total employment, $\underline{\bar{w}}\underline{L}$. It follows that $\underline{\bar{w}} = \frac{\underline{W}^{1+\psi}}{\underline{L}}$, and therefore, $\frac{d\ln \underline{\bar{w}}}{d\underline{w}} = (1+\psi)\frac{d\ln \underline{W}}{d\underline{w}} - \frac{d\ln \underline{L}}{d\underline{w}}$. Using (B-37), it follows that

$$\frac{d\ln \underline{\bar{w}}}{d\underline{w}} = \frac{1}{D}\left[\text{Term 1} + \text{Term 2}\right] > 0.$$

Therefore, if firm productivity has a Pareto distribution, an increase in the minimum wage increases the average wage but reduces total employment.