FAIR APPORTIONMENT AND THE BANZHAF INDEX

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1. Introduction. Justice Harlan's dissenting opinion in *Whitcomb* v. *Chavis* repudiated the Banzhaf index as a measure of voter power in apportionment decisions because of the supposed sensitivity of its calculations to minor variations in assumptions. Justice Harlan claimed such minor variations in assumptions could lead to differences in results on the order of magnitude of 120,000,000,000,000,000,000, but did not offer a rationale for his claim, the basis of which is not immediately obvious even to those familiar with the Banzhaf index and its properties. We provide formulae to determine Banzhaf power scores under varying assumptions as to voter partisan preferences and to verify Justice Harlan's calculations. We also briefly discuss the relevance of these formulae to a New York State Court of Appeals Decision, *Iannucci* v. *Board of Supervisors*, in which the Banzhaf index has been mandated as *the* test of the fairness of apportionment in county legislatures that make use of weighted voting.

2. Background. In three articles that appeared in American law journals in the mid-1960's, a lawyer named John Banzhaf III proposed to evaluate representation systems in terms of the extent to which they allocated "power" fairly [1], [2], [3]. Banzhaf's analysis makes use of game-theoretic notions in which power is equated with the ability to affect outcomes.

Consider a group of citizens choosing between two opposing candidates. To calculate the power of the individual voter, we generate the set of all possible voting coalitions among the district's electorate. If there are N voters in the district, then there will be 2^N possible coalitions. Then we ask, for each of these possible coalitions, whether a change in an individual voter's choice from candidate A to candidate B (or from candidate B to candidate A) would alter the electoral outcome. If so, that voter's ballot is said to be *decisive*. The (absolute) Banzhaf index of a voter's power is defined as the number of the voter's decisive votes divided by 2^N. The higher the percentage of voter coalitions in which a voter's vote is decisive, the higher that voter's power score. The Banzhaf index has considerable intuitive appeal; power is based on ability to affect outcome.

For single-member district systems (smds) whose districts are of equal population, all voters have identical power. But what about the case of multiple-member district systems (mmds), with districts of more than one size? Here, since the voters who elect k representatives have k times as much impact as voters who can elect only one representative, we might think that to equalize voter power we should assign to each district a number of representatives proportional to the size of the district's population since, intuitively, we would expect a voter's ability to decisively affect outcomes should be inversely proportional to district population. Banzhaf [2] pointed out that this argument is mathematically incorrect.

In a two-candidate/party contest where all voters have equal weight, in order for a voter to be decisive in a district of size N the rest of the voters (who are $N-1$ in number) must split half for one candidate/party and half against. A straightforward combinatoric analysis reveals ([2], [7], [6]; *Whitcomb* v. *Chavis* (1970) 403 U.S. at 145 n. 23) that, *if all combinations of vote outcomes are equally likely* (i.e., if each voter is equally likely to vote for either candidate/party), then the number of each member's decisive votes, b, is given by:

$$b=\frac{2(N-1)!}{\left(\frac{N-1}{2}\right)!\left(\frac{N-1}{2}\right)!}. \tag{1}$$

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We can examine the link between b and N by using Stirling's approximation [4], [2], [7], [6],

$$N! \approx e^{-N} N^N (2\pi N)^{1/2} \tag{2}$$

to rewrite (1) as

$$b \approx \frac{2^{N+1}}{(2\pi(N-1))^{1/2}}. \tag{3}$$

Thus, each member's Banzhaf index, which we shall denote by B_i, is simply

$$B_i = \frac{2\binom{N-1}{\frac{1}{2}(N-1)}}{2^N} \approx \frac{2}{(2\pi(N-1))^{1/2}}. \tag{4}$$

This analysis can be applied to electoral systems involving both single- and multi-member districts. We see from expression (4) that B_i is approximately inversely proportional to the square root of N district population. Thus, if we wish to assign all voters equal power to affect outcomes, we should assign each district a number of representatives proportional to the *square root* of district population, rather than directly proportional to district population.[1]

In a case decided in 1970, *Whitcomb* v. *Chavis*, 403 U.S. 143, the Supreme Court dealt directly with Banzhaf's concept of voter power. The case involved Indiana's scheme of single and multiple member districts for its state legislature. The plaintiffs, citing Banzhaf's work [2], argued that voters in the multiple member districts were overrepresented because citizens in the larger district had a power disproportionate to their population.

We can make this argument explicit as follows: if mmds (with population mN) elected m representatives, then each voter in such a district would have a power proportionate to $m/\sqrt{mN}$; while those in smds with a population of N would have a power proportional to $1/\sqrt{N}$. Since $m/\sqrt{m} > 1$, for $m > 1$, this would be denying to all citizens an "equally effective voice in the election of members of his legislature" (377 U.S. at 565).

The Banzhaf [2] argument was rejected decisively in *Whitcomb*, both in the majority opinion and in Justice Harlan's dissenting opinion. Only Harlan's opinion, however, dealt forthrightly with the intellectual merits of the Banzhaf argument.

In Justice Harlan's dissenting opinion in *Whitcomb* v. *Chavis* he pokes some more or less good-natured fun both at the Banzhaf index and at his brethren on the Court. First, Harlan (403 U.S. at 168, n. 2) cites the majority views on the Banzhaf index; to wit that, while mathematically correct, its implications can safely be ignored because it "does not take into account any political or other factors which might affect the actual voting power of the residents, which might include party affiliation, race, previous voting characteristics or any other factors which go into the entire political voting situation" (Ante at 145, 146). Then he retorts sarcastically that "precisely the same criticism applies with even greater force to the one man, one vote opinions of the Court. *The only relevant difference between the elementary arithmetic on which the Court relies and the elementary probability theory on which Professor Banzhaf relies is that calculation in the latter field can't be done on one's fingers*" (403 U.S. at 168 n. 2 [emphasis ours]).

Harlan then goes on to lampoon the absurdity of the Banzhaf index's simplifying assumptions e.g., that "the voting habits of all members of the electorate are alike" and that "each voter is equally likely to vote for either candidate before him" (403 U.S. at 168), and asserts that "minor variations in these assumptions can lead to major variations in results" (403 U.S. at 169). Harlan looks at a case involving 300,000 voters and claims:

If the temper of the electorate changes by one-half of one percent (more precisely, the result follows if the second of Professor Banzhaf's assumptions is altered so that the probability of each voter's selecting candidate A over candidate B is 50.5% rather than 50%) then each individual's voting power is reduced by a factor of approximately 1,000,000. Or if a few of the 300,000 voters are committed—say 15,000 to candidate A and 10,000 to candidate B—the probability of any individual's casting a tie-breaking vote is reduced by a factor on the rough order of 120,000,000,000,000,000,000. Obviously in comparison with the astronomical differences in voting power which can result from such minor variations in political characteristics, the effects of the 12% and 28% population variations considered in *Abate* vs. *Mundt* and in this case are *de minimis*.

Harlan does not indicate how he arrives at the figures 1,000,000 and 120,000,000,000,000,000,000. It is far from obvious, even to someone familiar with the Banzhaf index and its properties, where those numbers come from.

In this paper we shall investigate the sensitivity of Banzhaf power calculations to variations in assumptions about coalitional probabilities to see how results can vary as radically as Harlan claims they do. This issue is an important one, not merely because it will clear up the puzzlement of scholars reading Harlan's opinion in *Whitcomb* who are unable to determine on what basis his calculations are derived, but also because the Banzhaf index is enshrined into law as *the* test of fair apportionment of *weighted* voting systems in the state of New York. Thus, the properties of the Banzhaf index remain very much of interest, despite the rejection in *Whitcomb* of its applicability to the mixed single- and multi-member district case.[2, 3]

3. The Impact of Varying the Assumptions on Which Banzhaf Power Calculations Are Based. First, let us consider the case where there are N identical voters, each of whom votes for candidate A with probability p (where p need not equal $\frac{1}{2}$). A given voter is decisive when the remaining $N-1$ voters divide equally between the two candidates. For specified p the expected proportion of coalitions in which the voter will be decisive, which we shall call $B_{i_{(p)}}$, is given by

$$B_{i_{(p)}} = \frac{2\binom{N-1}{\frac{1}{2}(N-1)} p^{\frac{1}{2}(N-1)}(1-p)^{\frac{1}{2}(N-1)}}{2^N}. \tag{5}$$

The impact of a deviation from equiprobability in p varies as a function of N. In particular, the ratio of $B_{i_{(p)}}$ to $B_{i_{(1/2)}}$ is given by

$$\frac{B_{i_{(p)}}}{B_{i_{(1/2)}}} = (2p)^{\frac{1}{2}(N-1)}[2(1-p)]^{\frac{1}{2}(N-1)}. \tag{6}$$

To simplify calculation of this ratio it is convenient to take logarithms. Thus

$$\frac{B_{i_{(p)}}}{B_{i_{(1/2)}}} = \text{antilog}\left[\tfrac{1}{2}(N-1)(\log 2p + \log(2(1-p)))\right]. \tag{7}$$

For the 300,001 voter example considered by Harlan, for which $p=.505$, the ratio given in expression (7) is 3.1×10^{-7}.

Now, let us consider the case where N voters, of whom $K_1\left(\frac{1}{2}(N+1)>K_1\right)$ are committed to candidate A and $K_2\left(\frac{1}{2}(N+1)>K_2\right)$ to candidate B. For simplicity, let us initially assume $p=\frac{1}{2}$, i.e., that all coalitions of uncommitted voters are equally likely. A voter is decisive only when $\left(\frac{1}{2}(N-1)-K_1\right)$ of the (N_1-K_2-K-1) remaining undecided voters vote for A, i.e., when $\left(\frac{1}{2}(N-1)-K_2\right)$ of the remaining $(N-K_1-K_2-1)$ undecided voters vote for candidate B. There are $2^{N-K_1-K_2}$ possible voter coalitions, since K_1+K_2 of the voters are committed voters. The proportion of feasible coalitions in which the given voter will be decisive is given by

$$\frac{2\binom{N-K_1-K_2-1}{\frac{1}{2}(N-1)-K_1}}{2^{N-K_1-K_2}}. \tag{8}$$

Using Stirling's formula, and if we let $x=N-K_1-K_2-1$, $y=(N-1-2K_1)/2$, and $z=(N-1-2K_2)/2$, expression (8) can be approximated by

$$\frac{2e^{-x}x^x(2\pi(x))^{1/2}}{e^{-y-z}y^y z^{1/2(N+1-K_1-K_2)}\cdot 2\pi\sqrt{yz}\cdot 2^{N-K_1-K_2}}. \tag{9}$$

Using the approximations given in expression (9), the ratio of expression (8) to expression (4) can, after some algebra, be expressed as

$$\frac{((N-1)x)^{1/2}x^x}{2(yz)^{1/2}(2y)^y(2z)^z}. \tag{10}$$

In the case that $K_1, K_2 \ll N$ (read K_1, K_2 considerably less than N), we may approximate expression (10) by

$$\frac{\left(\frac{N-1}{x}\right)^{1/2}x^x}{(2y)^y(2z)^z}. \tag{11}$$

By expressing the formulae of (10) or (11) in terms of logarithms (cf. expression (7)) calculation of numerical results is straightforward. For the case where $N=300{,}001$, $K_1=15{,}000$, and $K_2=10{,}000$, we obtain a value of roughly 10^{-20}, as per Harlan's assertion.

In the special case that $K_1=K_2=K$, expression (9) directly simplifies to

$$\left(\frac{N-1}{N-1-2K}\right)^{1/2}. \tag{12}$$

Note also that for K_1, $K_2 \ll N$ we have

$$\left(\frac{N-1}{N-1-K_1-K_2}\right)^{1/2} \approx 1. \tag{13}$$

4. Conclusions. We have provided formulae that specify the impact on Banzhaf power calculations of deviations from the assumption of equiprobable coalitions and identical voters; and we have vindicated Justice Harlan's mathematics.

Acknowledgments. This research was supported by NSF Grant Soc. 77–24474, Political Science Program. It represents a continuation of work done jointly with Professor Howard Scarrow, State University of New York at Stony Brook, and came about because Professor Scarrow asked me how Justice Harlan came up with the figure of 120,000,000,000,000,000,000 given in *Whitcomb* v. *Chavis* (1976) 403 U.S. at 169. This paper was written to answer that question—a question that I'm sure has been asked (but rarely, if at all, answered) by everyone who has ever read Harlan's dissenting opinion in the *Whitcomb* case. I am indebted to the staff of the Word Processing Center of the School of Social Sciences, University of California, Irvine, for translating my scribbles and hand-written mathematical formulae into typed copy.

Notes

1. Such an assignment may violate other norms. If, for example, we assigned one representative for every 100 population in the square root of district size, then if there are 20,000 population spread equally over 2 smds, these voters (10,000 per district) would be entitled to have 2 representatives, 1 per district, since the square root of 10,000 is 100. Similarly, if there are 40,000 citizens spread equally over 4 smds (10,000 each) they would be entitled to 4 representatives. However, a single mmd of size 40,000 would be allocated only 2 representatives, since the square root of 40,000 is only 200. Thus, in this example, 20,000 voters would be entitled to as many representatives as 40,000 voters, the allocation of representatives to the 40,000 voters depends on how voters are divided among the districts.

2. The New York Court of Appeals in *Iannucci* v. *Board of Supervisors of the County of Washington* (1967) 282 N.Y.S. 2d 502 held:

> The principle of one man–one vote is violated when the power of a representative to affect the passage of legislation by his vote. . . does not roughly correspond to the proportion of the population in his constituency. Thus, for example, a particular weighted voting scheme would be invalid if 60% of the population were represented by a single legislator who was entitled to cast 60% of the votes. Although his vote would apparently be weighted only in proportion to the population he represented, he would actually possess 100% of the voting power whenever a simple majority was all that was necessary to enact legislation.

Similarly a plan would be invalid if it was *mathematically impossible* for a particular legislator representing say 5% of the population to ever cast a decisive vote. Ideally, in any weighted voting plan, it should be mathematically possible for every member of the legislative body to cast the decisive vote on legislation in the same ratio which the population of his constituency bears to the total population. Only then would a member representing 5% of the population have, at least in theory, the same voting power (5%) under a weighted voting plan as he would have in a legislative body which did not use weighted voting—e.g., as a member of a 20-member body with each member entitled to cast a single vote. This is what is meant by the one man–one vote principle as applied to weighted voting plans for municipal governments. A legislator's voting power, measured by the mathematical possibility of his casting a decisive vote, must approximate the power he would have in a legislature which did not employ weighted voting.

The *Iannucci* decision has had tremendous impact on New York County government, where 24 of 57 counties now use some form of weighted voting. In *Iannucci* the court held that counties would have to submit computer calculations of Banzhaf index scores to verify that any proposed legislature weight satisfied the *Iannucci* guidelines. Since weights assigned directly proportional to population represented often resulted in Banzhaf scores discrepant with weights (e.g., in a 3-member legislature with districts of size 2,000, 2,000, and 1,000 and a weight assignment of 2, 2, and 1, the Banzhaf scores of all 3 legislators are identical, despite the fact that one represents only half as many voters as the other two and has only half the weight they do), this has necessitated some counties' hiring professional assistance to generate weights that yield power scores concordant with population. Lee Papayanopoulous, a mathematician and computer programmer, has provided this service for well over a dozen New York counties in the past decade. (For a further discussion of the *Iannucci* decision and its effects on representation systems in the state of New York, see [5].)

3. The formulae we provide are relevant to challenges to the appropriateness of the Banzhaf index as a measure of *legislator* power, since in the legislature *even more than in the electorate* the assumptions of identical actors and equiprobable coalitions seem far indeed from the realities of politics; however, they must be adapted to deal with relative (normalized) Banzhaf power scores rather than the absolute (non-normalized) power scores discussed above. Moreover, for legislators, a smaller N leads to *far less* extreme variations in power as we vary coalitional probabilities.

Furthermore, unlike the U.S. Supreme Court in *Whitcomb*, which rejected Banzhaf calculations because of their lack of political realism, the New York Court of Appeals in *Iannucci* thought that this divorce from political realities was a positive feature of the Banzhaf index! In *Iannucci* actual voting patterns were held to be irrelevant. The sole criterion to be used in determining the constitutionality of a weighted voting scheme "is the mathematical voting power that each legislator possesses in theory—i.e., the indicia of representation—and not the actual voting power he possesses in fact—i.e., the indicia of influence" (20 N.Y. 2d at 252).

References

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MISCELLANEA

49. Who said *that*?

1. It will not occur to anyone to undervalue the merit of the mathematicians.
2. Mathematics talks about the things which are of no concern at all to man.

(Answers on p. 52.)