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Research note
The half-win set and the geometry of spatial voting games*

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Abstract. In the spatial context, when preferences can be characterized by circular indifference curves, we show that we can derive all the information about the majority preference relationship in a space from the win-set of any single point. Furthermore, the size of win sets increases for points along any ray outward from a central point in the space, the point that is the center of the yolk. To prove these results we employ a useful new geometric construction, the half-win set. The implication of these results is that embedding choice in a continuous n-dimensional space imposes great constraints on the nature of the majority-preference relationship.

In finite voting games knowledge of the majority-preference relation between some given alternative, a_i, and each of the remaining alternatives $a_j \epsilon A$ tells us nothing whatsoever about the directionality of majority preference between pairs in which a_i is not included, for example, between a_e and a_k. It might seem that imposing a spatial structure on alternatives would impose some constraints on the overall structure of majority preferences. But a remarkably strong result holds. If we know the geometry of the win set of any point x, then, when preferences are characterized by circular indifference curves, we can reconstruct the win-set of any other point in the space; that is, in the spatial context, if we know a single win-set, we can specify the complete structure of majority preference for the space; we need not know either the number of voters or the location of voters' ideal points.

Definition 1: The *win set of y*, denoted Win(y), is the set of alternatives $x \epsilon X$ such that xPy.

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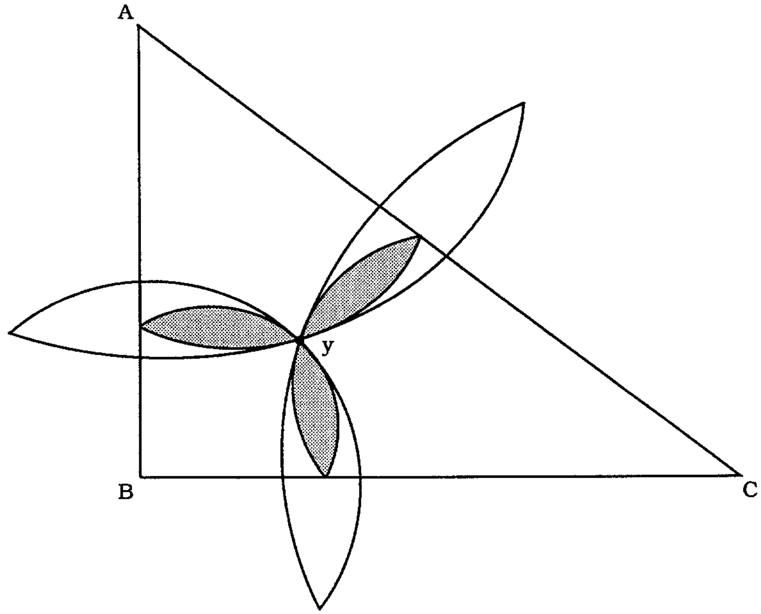

Figure 1. The petals of the win set and half-win set around y.

Definition 2: The *inverse win set of y*, denoted $\mathrm{Win}^{-1}(y)$, is the set of alternatives, $z \epsilon Z$, such that yPz.

Definition 3: The *half-win set* of a point, y, is the set of points which are obtained by uniformly reducing each ray in the win set by a factor of 1/2.

Definition 3 ′: The half-win set of a point, y, is the locus of intersections of rays from y perpendicular to the median lines in the space.

We provide an illustration in two dimensions (see Figure 1).

As far as we know, we are the first to identify the half-win set. It is a geometric construction with several nice properties.

Theorem 1: When voters' preferences are characterized by circular indifference curves, all the information about the majority-preference relationship in a space is contained within the (half-) win set of any point; that is, given a win set of any point, z, we can discover the win set of all other points.

Proof: Given Win(z), we may use the construction in Figure 2 to find whether xPy or yPx. Take the half-win set of any point, z, and the point which is the projection onto the xy line from a line parallel to the xy line through z at the point furthest out on z's half-win set on this parallel line. Since r is the projection of the median voter onto the xy line; if x is closer to r than is y, xPy; otherwise yPx.

A similar construction enables us to identify, for every line through a point,

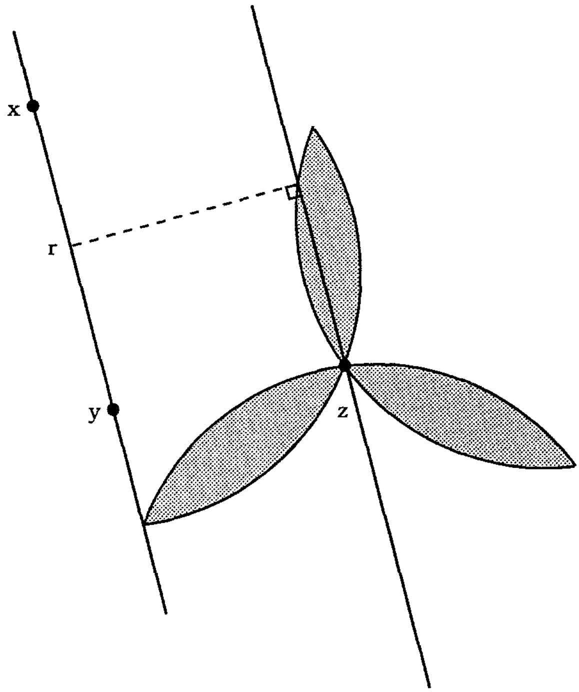

Figure 2. A construction to find whether xPy or yPx given knowledge of the half win set of z (shown shaded).

the furthest point on that line that is preferred to our starting point, and thus to trace out the complete win set of any point.[1]

We can also relate the half-win set to another important geometric construction, the yolk.

Definition 4: The yolk is the smallest sphere that intersects all median hyperplanes (McKelvey, 1986; Ferejohn, McKelvey, and Packel, 1984). In two dimensions the yolk is the smallest circle that intersects at median lines.

The yolk provides an upper bound on the uncovered set (Miller, 1977). If r is the radius of the yolk, then no point further than 4r from the center of the yolk can be uncovered (McKelvey, 1986; Feld, Grofman, and Miller, 1989; Feld et al., 1987). The size of the yolk also sets constraints on the limits of agenda manipulation (Feld, Grofman, and Miller, 1989; Miller, Grofman and Feld, 1989).

Theorem 2: The yolk is the minimum circle that encloses all the points in its center's half-win set.

Proof: Follows straightforwardly from the half-win set definition, because any circle that cuts all median lines also encloses the half-win set of the point at its center.

Theorem 2 shows the center of the yolk to be the point whose win set has the smallest radius. A natural question is how win sets vary in radius as a function of distance from o, the center of the yolk.

Definition 5: The *radius of a win set* (or *half-win set*) of a point is the radius of the smallest circle centered at the point that encloses the win set (half-win set) of the point.

Lemma 1: If x is directly between o (the center of the yolk) and some point, y, then the radius of the win set of x is less than the radius of the win set of y.

Proof: There must be a median line, 1, that is furthest away from x; the distance from x to 1 is the radius of the half-win set of x. The point o is closer to 1 than is x (else its half-win set would be larger than that of x); consequently, y is further from 1 than is x. The radius of the half-win set of y must be at least its distance to 1; therefore, its half-win set is larger than that of x. Therefore, of course, its win set also will be larger than the win set of x. Q.E.D.

Lemma 2: If y is directly between x and z, and if the radius of the win set of x is less than the radius of the win set of y, then the radius of the win set of z is greater than the radius of the win set of y.

Proof: Essentially identical to that of Lemma 1.

Lemma 3: For any line in the space, there is a point on that line with minimum win set radius, and the win set radius of all other points on that line monotonically increases in both directions from that minimum point.

Proof: Follows directly from the Lemma 2.

Definition 6: The locus of points that have win sets with an identical radius we shall call an *iso-radius locus* (or *iso-rad*, for short).

Theorem 3: All iso-radius loci (iso-rads) are convex surfaces surrounding the center of the yolk.[2]

Proof: Suppose an iso-rad was not convex; then, there would be a straight line that included three separate points of equal win set radius. This is contrary to Lemma 2. Q.E.D.

Figure 3 shows a three-voter illustration of iso-rads.

We believe that it is possible to prove an analogue to Theorem 3 for the "area of a win set." Our conjecture is that "iso-area" loci are convex surfaces surrounding the strong point (the Copeland winner). Grofman, Owen, Noviello,

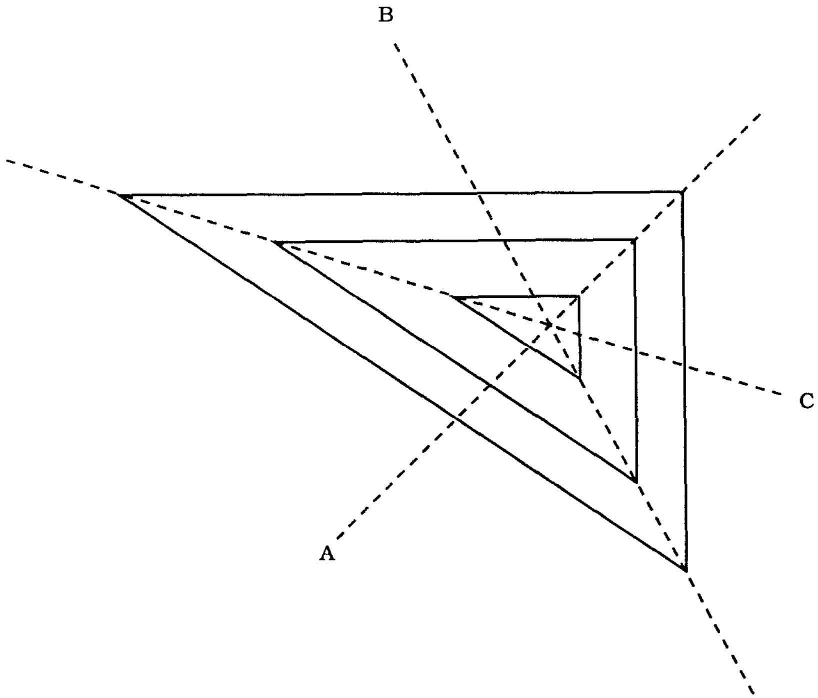

Figure 3. Iso-rads around the center of the yolk for a 3-voter example.

and Glazer (1987) demonstrate a result close to this, but not quite as strong.

The results we give here introduce a useful new geometric constraint, the half-win set, and show its relationship to one of the most important ideas in recent spatial social choice theory – the concept of the yolk (McKelvey, 1986; Feld et al., 1987). Our results demonstrate that the geometry of spatial voting games imposes powerful constraints on the nature of majority preference. If we know the win set of any point, we can specify the win set of all points. If we know that two points, x and y, lie along a line from the center of the yolk, if x is closer to the center of the yolk, then x must be majority preferred to more alternatives than is y. These results neatly complement the McKelvey result (McKelvey, 1986; Feld, Grofman, and Miller, 1989; Feld et al., 1987) that for any alternatives t and u, if r is the radius of the yolk, if the distance of u from the center of the yolk is more than 2r greater than the distance of t from the center of the yolk, then t P u.

Our results hold for the special case of Euclidean preferences (circular indifference curves), but a generalization of the concept of the yolk to the non-Euclidean case appears in Feld and Grofman (1987). Cox (1987) provides related results. Along the lines that these papers suggest, we believe that our results can be extended in a reasonably straightforward way to the non-Euclidean case. But, in the non-Euclidean case, rather than being able completely to specify win sets, knowledge of a single win set will impose outer and inner bounds on the location of the remaining win sets in the space.

Notes

1. In the special case where there is a Condorcet winner, that is, a point whose win set is of radius zero, Theorem 1 implies that, for any two points x and y, if y is further from the Condorcet winner than is x, xPy. Davis, DeGroot, and Hinich (1972) first proved this result. For a simple proof for Euclidean preferences, see Feld and Grofman (1987).
2. Notice that some portion of iso-rads are always straight lines.

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