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AVERAGE COMPETENCE, VARIABILITY IN INDIVIDUAL COMPETENCE, AND ACCURACY OF STATISTICALLY POOLED GROUP DECISIONS[1]

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Summary.—We correct propositions advanced by Sattler in 1966 on the accuracy of group judgments as a function of the mean competence and distribution of competence of group members. We provide the special distributions which, for fixed total group competence, minimize/maximize group majority rule accuracy. We also show how group majority competence can be above/below that of group mean competence.

Sattler (1966) has examined the effect of group variability on the accuracy of "statistically created" group decisions made by majority vote in "hypothetical" groups which do not actually meet and deliberate. He notes that a number of authors have found increasing group size increases the likelihood of a correct solution for such groups but asks "what happens if the group contains persons who differ greatly in their disposition to solve the problem correctly? Will the pooled group decision be better or worse than if the group were homogeneous with respect to their individual dispositions?" (Sattler, 1966, p. 676).

In answering this question, Sattler (1966, pp. 677-678) looks at three-members groups which differ in their average competence to reach a correct decision and in the variance of their competence distribution. To report his results concisely, we shall introduce some useful notation. Average group competence shall be denoted $\bar{p}$, $(0 \leqq \bar{p} \leqq 1)$, the competence of individual group members we shall denote p_i, and the majority vote accuracy of a group of size N we shall denote P_N.

Sattler (1966, p. 677) looks at three levels of competence ($\bar{p} > .5$, $\bar{p} = .5$, $\bar{p} < .5$) and three levels of variance (none, moderate, high). On the basis of his examination, he asserts four propositions:

(1) When $\bar{p} > .5$, $P_N > p$.

(2) For a $\bar{p} > .5$, the most heterogeneous group yields the highest value of P_N.

(3) For a $\bar{p} < .5$, the least heterogeneous group yields the highest value of P_N.

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(4) For $\bar{p} = .5$, group variability in competence does not affect P_N.

Unfortunately, Sattler's analysis is marred by a crucial failure, failure to control for the effects of symmetry. All the distributions he examined were symmetric around their mean. For an asymmetric distribution of competence, Result (4) is wrong. Furthermore, Result (2) is also wrong. For $\bar{p}$ fixed, the distribution of competence which maximizes P_N is *not* the distribution with highest variance, rather it is an asymmetric distribution of a rather special sort. Thus, two of Sattler's four propositions are in error. In this paper we present corrected results on the effect of variability in competence on group judgmental accuracy in the form of some very general theorems on the optimal distribution of competence. We also review other results on the effect of average competence and group size on P_N.

Review of Earlier Results

The basic theorem on the effect of pooling of judgments was stated by the French mathematician and philosopher, the Marquis de Condorcet (1785). This theorem, which can be thought of as a variation of the well-known "law of large numbers," was, however, "lost" for a number of years until rediscovered by Black (1958). [For a history of the theorem, see Grofman (1975).] The Condorcet Theorem in its original forms assumes that jurors are homogeneous, i.e., $p_i = p_j = p$ for all i, j. Hence, for homogeneous jurors we shall drop the subscripts.

Theorem I (Condorcet Jury Theorem)

If $1 > p > 1/2$, then P_N is monotonically increasing in N and $\lim_{N\to\infty} P_N \to 1$; if $0 < p < 1/2$, then P_N is monotonically decreasing in N and $\lim_{N\to\infty} P_N \to 0$; while if $p = 1/2$, then $P_N = 1/2$ for all N.

For a proof see Black (1958) or Grofman (1978). If $p > 1/2$, this theorem can be interpreted as "vox populi, vox dei." It is rather remarkable how fast P_N goes up (down) with N if $p > 1/2[p < (1/2)]$. We show results for $N = 1, 19$ in Table 1, taken from Grofman (1975).

Grofman (1978) generalizes this theorem to the case where the p_i are normally distributed with a variance equal to the binomial variance, with $\bar{p}$ replacing p in the expressions above. Grofman (1978) also looks at the question of when the group accuracy of a large group of not-so-smart people will be higher or lower than the P_N value for a smaller "blue-ribbon" group. He proves the following result.

Theorem II (Grofman Dummkopf-Witkopf Theorem)

For $p > .5$, a group of size $N + y$ each of whose members have competence $p - x$ is equivalent in judgmental competence to a group of size N whose members have judgmental competence p if and only if

TABLE 1*
PROBABILITY THAT A MAJORITY OF JURORS WILL REACH A CORRECT VERDICT FOR VARIOUS VALUES OF N AND p

N	.2	.4	.5	.6	.8
1	.2000	.4000	.5000	.6000	.8000
3	.1040	.3520	.5000	.6480	.8960
5	.0580	.3174	.5000	.6826	.9420
7	.0335	.2858	.5000	.7102	.9666
9	.0196	.2666	.5000	.7334	.9804
11	.0116	.2466	.5000	.7534	.9884
13	.0070	.2288	.5000	.7712	.9930
15	.0042	.2132	.5000	.7868	.9958
17	.0026	.1990	.5000	.8010	.9974
19	.0016	.1860	.5000	.8140	.9984

*N = group size, p = the probability that an individual member of the group will reach a correct judgment (Grofman, 1975).

$$y = N\{[.25x(2p - 1 - x)]/[p(1 - p)(p - x - .5)^2]\} \qquad [1]$$

Analogous results obtain if we replace p by $\bar{p}$, if the p_i are normally distributed and N is reasonably large. Expression [1] provides us a way of expressing trade-offs between p (or $\bar{p}$) and N in the form of "incompetence curves" (see Fig. 1). Grofman (1976) demonstrates the paradoxical result that it is sometimes possible to raise P_N by adding members to a group who actually lower the average competence of the group; the increase in N compensates

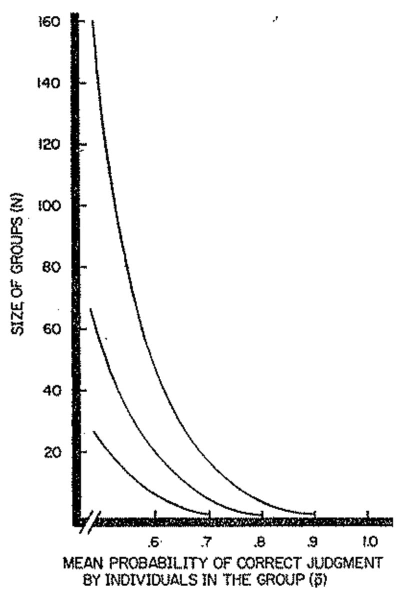

FIG. 1. Isocompetence curves for various values of mean group competence ($\bar{p}$). Source: (Grofman, 1975).

for the decrease in $\bar{p}$. This idea has been further examined in Margolis (1976), who notes that this paradox can occur only if the average competence of the added members is above .5.

Grofman (1978) also looks at the question of when the pooled group judgment can be expected to be more accurate than that of its *best* member. It is sometimes asserted that this can never occur (Einhorn, Hogarth, & Klempner, 1977, p. 168), but that is incorrect. The Grofman (1978) results on this question are, however, reported in the form of lengthy tables and do not permit ready summary; we refer the interested reader to the original source. For discussion of the expected distribution of competence of the "best" member of the group see Steiner and Rajaratnam (1961). For closely related models see Lorge and Solomon (1955) and Steiner (1966).

New Results

The key new result is one that tells us, for fixed $\bar{p}$ and N, how to assign competence so as to maximize (or minimize) P_N.

Theorem III (Optimal Distribution of Competence)[4]

If the sum total of competence is fixed (which sum we may arbitrarily denote as $\bar{p}N$), then P_N is maximized,[5]

(a) if $\bar{p}N \geqq (N+1)/2$, by setting a majority of the p_is to one;

(b) if $(N+1)/2 \geqq \bar{p}N \geqq N/2 - 0.2$, by setting $p_i = 0$ for $(N-1)/2$ members of the group and $p_j = \bar{p}[2N/(N+1)]$ for the remaining $(N+1)/2$ members of the group;

(c) if $\bar{p}N \leqq (N/2) - 0.4$, by setting $p_i = \bar{p}$ for all i.

Similarly, P_N is minimized

(a) if $N(1-\bar{p}) \geqq (N+1)/2$, i.e., if $1 > \bar{p}[2N/(N-1)]$, by setting a majority of the ps to zero;

(b) if $(N+1)/2 > N(1-\bar{p}) \geqq N/2 - 0.2$, by setting $p_i = 1$ for $(N-1)/2$ members of the group and $1 - p_j = (1-\bar{p})[2N/(N+1)]$ for the remaining $(N+1)/2$ members of the group;

(c) if $N(1-\bar{p}) \leqq N/2 - 0.4$, by setting $p_i = \bar{p}$ for all i.

The proof of this theorem is both complex and lengthy. It is available from the authors upon request.

[4]Proofs of Lemmas required for Theorems III and IV are on file in Document NAPS-03943 for \$4.00 for fiche or \$10.15 for photocopy from Microfiche Publications, P.O. Box 3513, Grand Central Station, New York, NY 10017.

[5]The values of 0.2 and 0.4 are only approximate. For intermediate values, the maximizing distribution is one which divides competence equally among exactly K members of the group, where K ranges from $(N+1)/2$ to N as we approach the bonds on the two inequalities. Analogous results obtain for the minimizing distribution. As N becomes large, the boundary conditions converge toward $\bar{p} = 1/2$ from above and below, but for small N differences obtain. For example, for $N = 3$, only when $\bar{p} = 9/16$ do the distributions of (a) and (c) yield identical values.

It is impotrant to note that the distribution which maximizes group competence for $p \geq 1/2$ is *not* that which maximizes variance, contra Sattler's claim. Indeed, for $\bar{p} > 1/2$ in the group competence maximizing distribution each "half" of the group is assigned *identical* p_i values (either 0 or $\bar{p}[2N/(N+1)]$.

By looking only at worst (best) cases, it is easy to generate two useful corollaries to Theorem III.

Corollary 1 to Theorem III.—A necessary condition for $P_N > 1/2$ is that

$$\{\bar{p}[2N/(N+1)]\}^{(N+1)/2} > 1/2 \ . \qquad [2]$$

A sufficient condition for $P_N > 1/2$ is that

$$\{(1-\bar{p})[2N/(N+1)]\}^{(N+1)/2} < 1/2 \ . \qquad [3]$$

Hence, a group can have $\bar{p} < 1/2$ and yet have $P_N > 1/2$. For example: (a) (.72, .72, 0); $\bar{p} = .48$, yet $P_N = .5184$. (b) (.8, .8, .8, 0, 0); $\bar{p} = .48$, $P_N = .512$. (c) (.8, .9, .7, 0, 0); $\bar{p} = .48$, $P_N = .504$. Similarly, a group can have $\bar{p} > 1/2$ and yet have $P_N < 1/2$. For example: (a) (1, .28, .28); $\bar{p} = .52$, yet $P_N = .4816$. (b) (1, 1, .2, .2, .2); $\bar{p} = .52$, yet $P_N = .488$.

It is possible to show that the above expressions are monotonic in N. We may look at the lowest odd value of N greater than 1, $N = 3$, to establish:

Corollary 2 to Theorem III.—A necessary condition for $P_N > 1/2$ is that $\bar{p} > \sqrt{2}/3 = .471$; while a sufficient condition for $P_N > 1/2$ is that $\bar{p} > [(3-\sqrt{2})/3] = .529$.

These corollaries suggest that, almost independent of distribution, if $\bar{p} > 1/2$, then $P_N > 1/2$; and we would suspect analogously that if $\bar{p} < 1/2$, then $P_N < 1/2$. Indeed, we can show an even stronger result.

Theorem IV (Distribution-Free Generalization of the Condorcet Jury Theorem)

If $\bar{p} < .5$, then $\lim_{N\to\infty} P_N \to 0$. If $\bar{p} > .5$, then $\lim_{N\to\infty} P \to 1$. If $\bar{p} = .5$, then

$$1 - e^{-1/2} < \lim_{N\to\infty} P_N < e^{-1/2} \qquad [4]$$

For example, for (.75, .75, 0), $\bar{p} = .5$ while $P_N = .5625$; for (5/6, 5/6, 6/6, 0, 0), $\bar{p} = .5$, yet $P_N = .5787$; while for (1, .5, 0), $\bar{p} = .5$ and $P_N = 1/2$. Hence, as asserted earlier, Sattler (1966) was in error in claiming that for $\bar{p} = 1/2$, the distribution of p_i values is irrelevant to P_N.

We shall conclude this section of our paper with one other useful result, which we may state metaphorically as "More heads are better than fewer, as long as they are good heads." For heterogeneous groups, if $p_i > .5$ for all i, then the greater the size of the majority in favor of an alternative, the more likely is that alternative to be the correct choice. This result need not obtain, however, if competence is unequally distributed even if $p > .5$. Consider the distribution (.8, .8, 0). If exactly two voters are in agreement, they are correct with probability 2/3. If all three voters are in agreement, they are correct with probability zero (Grofman, Owen, & Feld, in press).

We have corrected and considerably generalized the Sattler (1966) results. In particular we have fully specified the relationships between $\bar{p}$, N, and P_N and have specified, for fixed $\bar{p}$ and N, the distributions of individual competence which will maximize/minimize group accuracy. We believe these formal results will be of considerable value in analyzing the outcomes of experiments on the problem-solving abilities of groups of different sizes and different distributions of competence (Lorge, *et al.*, 1958).

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