Soc Choice Welfare (1990) 7: 71–74

Social Choice and Welfare

A Theorem Connecting Shapley-Owen Power Scores and the Radius of the Yolk in Two Dimensions

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The most common approaches to measuring pivotal power, the Shapley-Shubik value and the Banzhaf-Coleman index, can be interpreted as assuming equiprobable permutations of voter preferences and equiprobable combinations of voter preferences, respectively (Straffin 1977). Recent work on power indices (Shapley 1977; Owen 1971; Grofman et al. 1987; Shapley and Owen 1989) has proposed to measure power in spatial voting games in terms of more realistic assumptions about which coalitions will form. In two dimensions, we can posit that the only coalitions that will form are those that are connected with respect to a given line of cleavage and then consider what happens when we rotate the axis of cleavage, through the space. In effect we measure pivotal power in terms of the proportion of time that each voter can be expected to be the median voter. Straffin (forthcoming) refers to this approach as generating Shapley-Owen power scores, and we will use that label. This approach has been used by a number of authors to study power in various empirical contexts including cabinet coalition formation (Rapoport and Golan 1985), the electoral college (Rabinowitz and McDonald 1986), and party support groups (Grofman et al. 1988).

Another recent line of research in spatial voting games involves the concept of the yolk (Ferejohn et al. 1984; McKelvey 1986; Feld et al. 1987, 1988; Feld and Grofman 1988; Miller et al. 1989). In two dimensions, if voters have Euclidean preferences, the yolk is the smallest circle that touches all median lines. It is known that the size of radius of the yolk, r, sets bounds on majority rule win sets. In particular, for two points x and y, at distance d_x and d_y, respectively, from the center of the yolk, if $d_y+2r>d_x$, then a majority of voters must prefer x to y (Feld et al. 1987; McKelvey 1986).

In this note we connect the size of the yolk to bounds on the Shapley-Owen power scores that can be assigned to voters at a given distance from the center of the yolk. This is the first linking of which we are aware of research in these two traditions.

Theorem. *In two dimensions, where vote preferences are Euclidean, an actor with voter ideal point, x, a distance of d from the center of the yolk, can have a (normalized)*

Shapley-Owen power score, P, no greater than $\frac{2\arcsin\left(\frac{r}{d}\right)}{\Pi}$, *where r is the radius of the yolk.*

Proof. An actor can be pivotal only if it is on a median line. All median lines pass through the yolk; thus the only angles at which the actor can be pivotal are described by lines through the yolk. These range from one line tangent to the yolk to another at the opposite sides of the yolk. See Fig. 1.

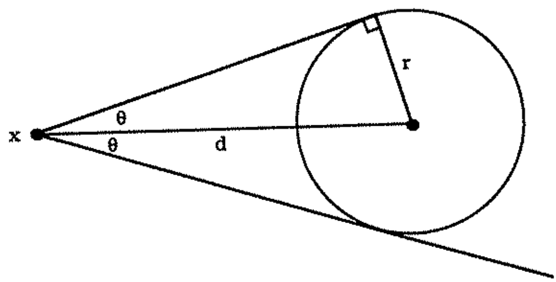

Fig. 1. Construction used to prove Theorem 1

Table 1. Bounds on Shapley-Owen power scores for a point x at a distance d from the center of the yolk (d expressed in integer yolk radii)

$d = \mathrm{K}r$	$\frac{2\arcsin\frac{r}{d}}{\pi}$
1	1.000
2	0.333
3	0.216
4	0.160
5	0.128
6	0.107
7	0.090
8	0.080
9	0.070
10	0.060
11	0.060
12	0.050
13	0.050
14	0.040
15	0.040
16	0.040

They span an angle 2θ as shown in Fig. 1. Since tangents are perpendicular to a radius; it follows that

$$\sin\theta = \frac{r}{d'}$$

and

$$\theta = \arcsin\left(\frac{r}{d}\right).$$

or

$$2\theta = \arcsin\left(\frac{r}{d}\right).$$

The power is that angle divided by the total, i.e.,

$$2\theta/\pi = \frac{2\arcsin\left(\frac{r}{d}\right)}{\pi}. \quad \text{Q.E.D.}$$

We show in Table 1 the nature of these bounds for values of d from one yolk radius to 16 yolk radii. As we move away from the yolk the maximal power of an actor declines (at a declining rate). Even two yolk radii away, an actor can have no more then one-third of total (Shapley-Owen) pivotal power.

Unfortunately this result holds only in two dimensions.

Conclusions

The link derived in Theorem 1 between centrality (relative to the yolk) and power (defined in Shapley-Owen terms) has important political implications. For example, if the policy is composed of interest groups, then any interest group whose ideal point falls far from the political "center" (defined as the center of the yolk) can have very little power in Shapley-Owen terms. Power is not only a function of the number of votes an interest group has, but it is also a function of its position. Centrally located interest groups have more power than peripheral ones.

In terms of policy positions, Shapley-Owen power refers to the power to move an outcome from one median line to another–shifts that are not likely to make much difference to the actor who is pivotal. However, if an actor can exact rents for being the pivotal actor, then these differences are important, and centrally located actors may be in the best position to derive non-policy benefits from having this ability to shift outcomes by a shift of their vote.

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