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PROBABILITY AND LOGIC IN BELIEF SYSTEMS*

ABSTRACT. This paper seeks to develop a formal (mathematical) model of belief systems based on the axioms of probability theory and propositional logic. By a belief system we mean a set of propositions along with an actor's objective probability assignments to (beliefs in) them, together with the relationships among and between propositions and beliefs. Belief systems are regarded as being comprised of interrelated elements.

In the paper are developed measures of the distance between sets of beliefs; of the congruence, coherence, and consistency of belief systems; and of the degree of polarization of belief systems – which are derived from one basic operation, symmetric difference.

We show that these measures possess a number of useful and powerful mathematical properties. Also, a model is set forth by which, from an actor's subjective probability assignment to propositions and pairwise conjunctions of propositions, we may then impute to the actor subjectively perceived truth functional relationships between propositions.

The potential uses and practical difficulties with the approach taken in the paper are also discussed, and the assertion is made that the measures developed enable us to simply distinguish between certain notions (e.g., congruence, consistency, coherence) too often and easily confused, provide us with the possibility of interval (or at least, quasi-interval level) measurement of certain properties of individual belief systems, and also allows us to make comparisons between the structures of different actors' belief systems.

Central to the research design of this paper is the notion that belief systems comprise interrelated elements and that changes in some elements of the system can be expected to (eventually[1]) effect changes in other elements of the system. Our aim is to present a measurement model which will make these linkages among beliefs and changes therein subject to precise empirical measurement.

We shall denote individual propositions by lower case letters, and sets of propositions by capitals. We shall denote by $a_1, a_2, \ldots, a_n$ the propositions in a set A which may be conjectured to be part of some belief system A'. We shall use the familiar logical operators $\rightarrow$, 'implies'; $\leftrightarrow$ 'equivalent to'; – negation'; $\wedge$ 'and'; $\vee$ 'or' to represent relationships (among propositions) as perceived by some given actor(s). Similarly, we shall use the set theoretic operators $\cup$, union; $\cap$ intersection; $\in$, membership in; $\subset$, is included in; $\supset$, includes; to represent relationships among sets.

Let us denote by $p(a_i)$ the (subjective) probability that an actor assigns to

Theory and Decision 4 (1973) 179–195. *All Rights Reserved*

a proposition a_i.[2] If we confine ourselves to propositions a_i containing free (unquantified) variables,[3] and expressing set theoretic statements such as "$x \in Y$, "$x \in Z$, etc., then $p(a_i)$ may be interpreted in frequency terms as the percentage of $x \in X$ which are perceived by the actor also to belong to Y. In general, a_i will then take the form $x \in X \rightarrow x \in Y$. If $a_i \equiv x \in Y$, then the reference class X is taken to be universal, i.e. $X = X \cup \bar{X}$. For example, the actor might be asked to assess the probability that members of the John Birch Society support (i.e. are included in the class of those who support) school busing for the purpose of integration. In this case, $p(a_i)$ would be the percentage of John Birch members whom the actor believes also to be supporters of schools busing for integration, (i.e. $p(a_i)$ gives the likelihood of a randomly chosen Bircher also supporting busing). Similarly, we invite the reader to interpret $p(a_i \wedge a_j)$ as the percentage of x's for which the relationships a_i('$x \in Y$') and a_j('$x \in Z$') are believed by the actor to simultaneously hold. Thus, our model will be restricted to beliefs about particular categories of actors or events. A number of difficulties occur in coping with subjective probabilities of 'unique' events (Nagel, 1939, esp. Chapter III). Although we believe the difficulties are not insurmountable, we shall not attempt to deal with the subjective probability of propositions such as 'Richard Nixon will be regarded by history as a good President,' but shall confine ourselves to propositions which state set theoretic relationships which can be interpreted in percentage terms such as that previously cited about John Birch Society members.

Consider now some set of propositions which are conjectured to be elements of a given actor's belief system. If these propositions are perceived by the actor as related, i.e. if some are perceived as implied (or, more loosely, evoked) by, or are perceived of to imply (evoke) others, then the actor's belief in one of these propositions 'ought' logically to be constrained by his belief in other propositions which he perceives as related. If the actor's (subjective) probability assignments satisfy the usual probability axioms:

(1) I. $0 \leqslant p(a_i) \leqslant 1$

II. $p(a_i \vee \bar{a}_i) = 1$

III. $p(a_i \wedge \bar{a}_i) = 0$

IV. $p(a_i \vee a_j) = p(a_i) + p(a_j) - p(a_i \wedge a_j)$

V. $p(a_i \wedge a_j) = p(a_j \mid a_i)\, p(a_i)$, where $p(a_j \mid a_i)$ refers to the conditional probability of a_j given a_i,[4] and if the actor is 'logical' in the sense that his (subjective) beliefs about the logical relationships between propositions obey the rules of propositional logic,[5] then we may readily prove a number of theorems which indicate the constraints imposed upon an actor's beliefs when some aspects of his belief system are assumed fixed.

For example let us assume that $a_1 \rightarrow a_2$ (i.e. the actor *subjectively* perceives the first proposition to imply the second), then $p(a_1)$ 'ought' to be less than or equal to $p(a_2)$[6] and $p(a_1 \wedge a_2)$ 'ought' to equal $p(a_1)$. Similarly, if $a_2 \wedge a_1 \rightarrow a_3$ and a_1 and a_2 are perceived as unrelated (i.e. independent in the sense that knowledge about one proposition is not perceived by the actor as providing him with knowledge about the other[7]), then $p(a_3)$ 'ought' to be greater than or equal to $p(a_1)\cdot p(a_2)$. These and similar results, although independently derived by the senior author, were subsequently found in Reichenbach (1938, Section 35). Related results may be also found in Rescher (1969). A related technique for determining the logical implication of given *changes* in an actor's beliefs is given by McGuire (1960c), but McGuire has never fully developed the implications of his model and his presentation is marred by some technical errors which he has acknowledged (McGuire, 1968).

Let us now consider a simple verbal example. Suppose that a man believes that all Americans who are not members of the John Birch Society are communist sympathizers and also believes that no member of the John Birch Society is a communist sympathizer. We can ask him his assessment of the percentage of Americans who are members of the John Birch Society and his assessment of the percentage of Americans who are communist sympathizers. Now, if the man is 'consistent,' in one clear sense of that term[8], then these two percentages should sum to 100% (the subjective probabilities should sum to one.) We are *not* requiring that the man's assignment of probabilities reflect any objectively determined 'accurate' probabilities. He may feel, for example, that there are proportionally twice as many Birch members as in fact there really are, but as long as his estimate is reflected in a correspondingly reduced estimate of the proportion of communist sympathizers, he is consistent. Moreover, we might also choose to call him "rational". Rationality, so defined, would have to do with the consistency of relationships between elements of a belief system, *not* with the "correct" assignment of probabilities to, or the empirical accuracy of perceived logical relationships among the elements themselves. Rationality so defined is a property of systems of (three or more) beliefs, not of beliefs themselves. Thus, a man might be said to be unreasonable (paranoid, euphoric, pessimistic, stupid, or ahead of his times) if his subjective probability assessments differ significantly from those held to be correct, but in our terms he *may* still be perfectly 'rational'. Similarly, to believe that all non-Birchers are Communist sympathizers, and conversely, might from one perspective be called 'irrational'. In our terms, however, this equivalence relationship is simply subjective rather than objective, and not 'right' or 'wrong', 'rational' or 'irrational.'

We shall be interested in obtaining precise measures for the subjective relationships among elements of a belief system. (Throughout we shall assume that the experimenter possesses whatever experimental data are required for our measures to be determinate).

Consider the following set theoretic operator $\oplus$, the symmetric difference operator. Let a_1, a_2 be sets, then

$$(2) \qquad a_1 \oplus a_2 \underset{\text{df}}{\equiv} (a_1 \vee a_2) - (a_1 \wedge a_2)$$

The operator $\oplus$ gives us the elements in the sets a_1, a_2 which are unique to them, exclusive of the elements they hold in common; it is the exclusive sense of the word 'or'.

For sets a_1 and a_2, $a_1 \oplus a_2$ may be regarded as a measure of the difference between them, given by the hatched area in Figure 1.

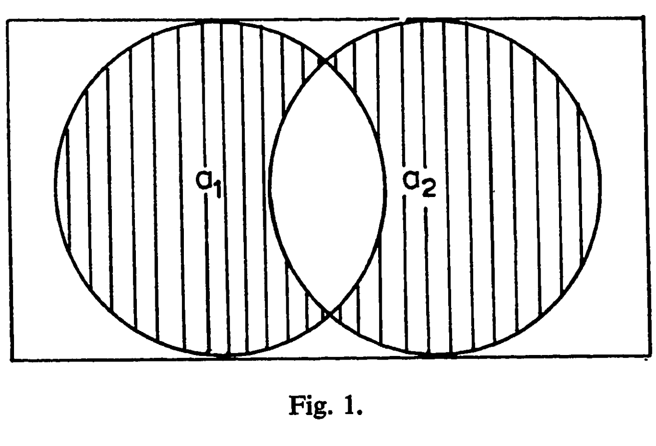

Fig. 1.

Now, let us consider a measure function m, defined on the sets such that:

(3) I. $m(\emptyset) = 0$, where $\emptyset$ is the empty set
II. $0 \leqslant m(a_i)$ for any a_i.
III. If $a_i \wedge a_j = \emptyset$, then
$m(a_i \vee a_j) = m(a_i) + m(a_j)$.

This measure function may be interpreted in a number of different ways. For example, $m(a_i)$ may be taken as the 'size' of a_i. We shall, however, be concerned with only one interpretation of $m(a_i)$, that in which a_i is taken to be a proposition, and $m(a_i)$ is interpreted as the actor's belief in a_i's being true.

We may readily show that if p (subjective probability) satisfies the usual probability axioms, as hypothesized above, then p will be a measure function. Now, consider:

(4) $$p(a_1 \oplus a_2) \underset{\text{df}}{\equiv} p(a_1 \vee a_2) - p(a_1 \wedge a_2).$$

We can readily see that

(5) $$p(a_1 \oplus a_2) = p(a_1) + p(a_2) - 2p(a_1 \wedge a_2).$$

Let us define, for notational convenience,

(6) $$P \oplus (a_i, a_j) \underset{\text{df}}{\equiv} p(a_i \oplus a_j)$$

We may readily establish that $p \oplus (a_i, a_j)$ is a superadditive distance metric, i.e.

(7) I. $p \oplus (a_i, a_j) \geqslant 0$ and $p \oplus (a_i, a_j) = 0$ if and only if $a_i \leftrightarrow a_j$
II. $p \oplus (a_i, a_j) = p \oplus (a_j, a_i)$
III. $p \oplus (a_i, a_j) + p \oplus (a_j, a_k) \geqslant p \oplus (a_i, a_k)$.

A proof of this result is given in Restle (1961) and in Majone and Sanday (1971). See also Restle (1959).

The symmetric difference operator gives us, in effect, the measure of commonality of 'subjective meaning' for two propositions, viz. a measure of the extent to which they are subjectively seen to imply and be implied by the same things. The greater their commonality, the more they are seen to have common implications and antecedents; the fewer their differences, the greater their degree of 'andness' and the smaller their degree of 'orness,' the less they are seen as incongruent. In those cases where there is complete commonality between a_i and a_j, where the subjective implications (evocations) of the two propositions are identical, $p \oplus (a_i, a_j)$ will be zero. Similarly, where there is no commonality between them and where they are negations of one another in their subjective implications, $p \oplus (a_i, a_j)$ will be maximal, in this case equal to one. (See Table I). Other authors who make use of the symmetric difference operator in this context are Hays (1959), and Bruner *et al.* (1959).

TABLE I

Hypothesis relating a_i and a_j		Value of $p\oplus(a_i, a_j)$ associated with h_i being true
h_1	$a_i \leftrightarrow a_j$	0
h_2	$a_i \rightarrow a_j$	$p(a_j) - p(a_i)$
h_3	$a_j \rightarrow a_i$	$p(a_i) - p(a_j)$
h_4	a_j	$1 - p(a_i)$
h_5	a_i	$1 - p(a_j)$
h_6	$\overline{a_i} \rightarrow a_j$	$2 - p(a_i) - p(a_j)$
h_7	$a_i \rightarrow \overline{a_j}$	$p(a_i) + p(a_j)$
h_8	$a_i \wedge a_j \rightarrow a_j \wedge a_i$	$p(a_i) + p(a_j) - 2p(a_i, a_j)$

Column 2 of Table I gives the values which $p \oplus (a_i, a_j)$ 'ought' to take on iff a_i and a_j are perceived to be related as specified in column 1. Table I is exhaustive of the eight basic pure truth functional relationships between two propositions; the remaining eight may be generated as negations of the hypotheses in the table. Of the values of $p \oplus (a_i, a_j)$ for the sixteen possible cases, four are of particular interest.

(8) $h_1 \quad (a_i \leftrightarrow a_j) \leftrightarrow (p \oplus (a_i, a_j) = 0)$

$h_2 \quad (a_i \rightarrow a_j) \leftrightarrow (p \oplus (a_i, a_j) = p(a_j) - p(a_i))$

$h_3 \quad (a_j \rightarrow a_i) \leftrightarrow (p \oplus (a_i, a_j) = p(a_i) - p(a_j))$

$h_{16} \quad (a_i \leftrightarrow \overline{a_j}) \leftrightarrow (p \oplus (a_i, a_j) = 1)$

We may use these relationships to define subjective equivalence, implication and negation. Thus, by ascertaining an actor's subjective probability assignments to a_i, a_j and $a_i \wedge a_j$, we may then impute to him the logical relationship between a_i and a_j to which his computed value of $p\oplus(a_i, a_j)$ most nearly corresponds. Thus in the limiting cases, if $p\oplus(a_i, a_j)=0$, then we shall say a_i is subjectively equivalent to a_j; if $p\oplus(a_i, a_j)=p(a_j)-p(a_i)\neq 0$, then we shall say a_i subjectively implies a_j; if $p\oplus(a_i, a_j)=1$, then we shall say that a_i and a_j are subjectively negations of one another. Analogously, we may use the symmetric difference operator to define (subjective) logical independence, since if two propositions are (subjectively) independent, then $p\oplus(a_i, a_j)$ 'ought' to take on the value $p(a_i)+(a_j)-2p(a_i)p(a_j)$. (See note 7). We shall refer to the hypothesis that two propositions are (subjectively) independent as h_{I}.

In effect, then, by knowing only an individual's direct probability assignments and simple pairwise probability assignments among propositions, we can in theory impute to him an entire belief system complete with perceived relationships among propositional elements. Moreover, for each truth functional relationship possible between two propositions we can construct a measure of the extent to which the experimental value of $p\oplus(a_i, a_j)$ approximates any such hypothesized relationship; and, therefore, the extent to which the imputed relationship may be said to obtain. For convenience, we wish a measure which varies between 0 and 1; and is 0 if and only if the hypothesized relationship between propositions is perfectly exemplified by the experimental value of $p\oplus(a_i, a_j)$, and is 1 if and only if the experimental value of $p\oplus(a_i, a_j)$ is as far from its hypothesized value as it can be given the experimental values of $p(a_i)$, $p(a_j)$ and $p(a_i \wedge a_j)$. Let $p\oplus(h_i, a_i, a_j)$ be defined as that value of $p\oplus(a_i, a_j)$ which obtains when the relationship h_i holds between propositions a_i and a_j. Note that the values of $p\oplus(h_i, a_i, a_j)$ for h_1 thru h_8 are given by column 2 of Table I. A measure which satisfies our requirements is

$$(9) \qquad d_{h_i}(a_i, a_j) \underset{\mathrm{df}}{\equiv} \frac{|p\oplus(a_i, a_j) - p\oplus(h_i, a_i, a_j)|}{\max\langle p\oplus(h_i, a_i, a_j), 1 - p\oplus(h_i, a_i, a_j)\rangle}.$$

We may readily verify that $d_{h_1}(a_i, a_j) = p\oplus(a_i, a_j)$.

By means of $d_{h_i}\,({}_{h_1}, a_j)$ we can measure the fit between an individual's views as to the logical relationship between two propositions as he might state it when asked directly, and the relationship between them that could be inferred from the data provided by his probability assignments. More importantly, from probability data alone we can infer (psychological) connections between propositions. In order to see this link in more familiar terms, let us express experimentally obtained subjective probabilities in terms of the entries in a 2 × 2 contingency table (Table II).

TABLE II

	a_i	a_i
a_j	a	b
$\bar{a}_j$	c	d

where $a+b+c+d=100$ and where

$$(10)\qquad p \oplus (a_i, a_j) = p(a_i) + p(a_j) - 2p(a_i \wedge a_j) = \frac{(a+c)+(a+b)-2a}{100} = \frac{b+c}{100}$$

We may restate the relationships of Table I in contingency table terms as in Table III.

Similarly, we may show that[9]

$$(11)\quad d_{h_I}\ (a_i, a_j) = \frac{2ad - 2bc}{ab + ac + b^2 + bd + c^2 + cd + 2ad}\quad \begin{cases} 0 & \text{iff } ad = bc \\ 1 & \text{iff } b + c = 0 \end{cases} \quad \text{for } ad \geqslant bc$$

$$(12)\quad d_{h_I}\ (a_i, a_j) = \frac{2bc - 2ad}{a^2 + ab + ac + bd + cd + d^2 + 2bc}\quad \begin{cases} 0 & \text{iff } ad = bc \\ 1 & \text{iff } a + d = 0 \end{cases} \quad \text{for } ad \leqslant bc$$

TABLE III[a]

	Hypothesis relating	$d_{h_i}(a_i, a_j)$	
h_1	$a_i \leftrightarrow a_j$	$\frac{b+c}{100}$	$\begin{cases} 0 \text{ iff } b+c=0 \\ =1 \text{ iff } a+d=0 \end{cases}$
h_2	$a_i \rightarrow a_j$	$\frac{c}{a+c+d}$	$\begin{cases} =0 \text{ iff } c=0 \\ =1 \text{ iff } a+d=0 \end{cases}$
h_3	$a_j \rightarrow a_i$	$\frac{b}{a+b+b}$	$\begin{cases} =0 \text{ iff } b=0 \\ =1 \text{ iff } a+d=0 \end{cases}$
h_4	a_j	$\frac{c}{a+c}$	$\begin{cases} =0 \text{ iff } c=0 \\ =1 \text{ iff } a=0 \end{cases}$
h_5	a_i	$\frac{b}{a+b}$	$\begin{cases} =0 \text{ iff } b=0 \\ =1 \text{ iff } a=0 \end{cases}$
h_6	$\bar{a}_i \rightarrow a_j$	$\frac{d}{b+c+d}$	$\begin{cases} =0 \text{ iff } d=0 \\ =1 \text{ iff } b+c=0 \end{cases}$
h_7	$a_i \rightarrow \bar{a}_j$	$\frac{a}{a+b+c}$	$\begin{cases} =0 \text{ iff } a=0 \\ =1 \text{ iff } b+c=0 \end{cases}$
h_8	$a_i \wedge a_j \leftrightarrow a_j \wedge a_i$	0	

[a] Formulas shown for $p\oplus(h_i, a_i, a_j) \leqslant 1 - p\oplus(h_i, a_i, a_j)$.

So far our use of the symmetric difference operator and of measures based upon it have been applied only to pairwise conjunctions of propositions. But cognitive systems in general (and ideologies in particular) consist not of pairs of propositions, but of systems of propositions, and hence of sets of propositions. We must therefore seek to expand the domain of our measures. Let us, therefore, for two sets of propositions, define the distance between them as the cartesian product of the distances between their propositional elements, appropriately normalized so as to remain within the interval [0, 1]. Utilizing the cartesian product maintains the distance metric properties of $p\oplus$.

Let A, B be sets of propositions of m and n elements respectively. Let

$$(13) \qquad p\oplus(A, B) \underset{\text{df}}{\equiv} \frac{\sum_{a_i \in A} \sum_{b_j \in B} p\oplus(a_i, b_j)}{m \cdot n} \underset{\text{df}}{\equiv} d_{h_1}(A, B);$$

$p\oplus(A, B)$ can be regarded as a measure of the distance between sets of beliefs.

Equivalently, we shall let

$$(14)\quad d_{h_i}\quad (A, B) \underset{\text{df}}{\equiv} \frac{\sum_{a_i \in A} \sum_{b_j \in B} d_{h_i}(a_i, b_j)}{m \cdot n}.$$

Some of these general measures will prove useful when we consider certain properties of belief systems, such as coherence, below.

Finally, we wish to know, loosely speaking, which pairs of propositions drawn from two sets of propositions 'lie closest to each other or have the most in common.' Such propositional pairs may be regarded as potential links or bridges between two otherwise incongruent sets of beliefs.

Let us define

$$(15)\qquad L(A, B) \underset{\text{df}}{\equiv} \min_{\substack{a_i \in A \\ b_j \in B}} \{p \oplus (a_i, b_j)\}.$$

L is then the minimum (shortest) distance between two sets of beliefs. It should be clear that the pair of propositions for which $p\oplus$ is minimum need not be unique. All such elements may be regarded as 'access points' or 'links' between the sets. Since for any two sets there will exist a continuum of pairs of propositions from both sets of propositions from minimally to maximally distant from one another, there will be a continuum of more or less 'accessible' linkage points between the two sets of beliefs. If we wish to 'transmute' belief systems, common sense suggests that the points from which to begin the transformation are the beliefs which are shared – and our model provides a precise way of identifying those beliefs.

Let us now consider the notion of the 'consistency' of a belief system. We may differentiate at least four meanings which can be attached to the term: (1) the actor's beliefs are consistent with scientific truth (the facticity of the real world as somehow 'objectively' rather than 'subjectively' determined); (2) the actor's beliefs are logically consistent, i.e. the beliefs satisfy the axioms of propositional logic and the axioms of subjective probability (this is the sense in which we were using consistency earlier); (3) the actor's beliefs are systematically interrelated, they 'hang together.'; (4) the actor's beliefs are not randomly generated and are stable at least in the short run (this type of consistency is related to the notion of construct reliability common in the sociological literature).

We shall label these types of consistency, respectively, 'objective consistency', logical consistency', 'connectedness' ('systemicity') and 'reliability'. These types of consistency are not independent of each other. For example, if the actor's beliefs are 'objectively' consistent, they are consistent in all of the other senses, at least under most commonly held epistemological and ontological views. Indeed, each type of consistency implies consistency of a lower type.

We may readily develop measures of the extent to which a belief system exemplifies a particular level of consistency. We shall present such measures for the connectedness and reliability of systems of beliefs; these measures will vary between 0 and 1.

To measure type 3 consistency, connectedness (systemicity), we make use of the measure defined on the hypothesis 'a_i and a_j are perceived of as independent', that is, we let

$$(16)\qquad \text{Connectedness}(A') \underset{\text{df}}{\equiv} \frac{|1-\sum_{a_i}\sum_{a_j} d_{h_1}(a_i, a_j)|}{m(m-1)}, \quad a_i \neq a_j.$$

We may do so since independence, as we have defined it, may be regarded as the paradigmatic case of unsystemicity.

To measure type 4 consistency, reliability, we may obtain for a given actor the values $p(a_i)$, $p(a_i \wedge a_j)$ and also, say, $p(a_i \vee a_j)$. With this information we may determine $p\oplus(a_i, a_j)$ in more than one way. For example, given the values of $p(a_i)$, $p(a_j)$, $p(a_i \wedge a_j)$, and $p(a_i \vee a_j)$ we could determine $p\oplus(a_i, a_j)$ either as $p(a_i)+p(a_j)-2p(a_i \wedge a_j)$ or as $p(a_i \vee a_j) - p(a_i \wedge a_j)$. We could then define the reliability of an actor's beliefs A' as

$$(17)\qquad \text{Reliability}(A') \underset{\text{df}}{\equiv}$$

$$\underset{\text{df}}{\equiv} \frac{\sum_{a_i}\sum_{a_j} \left| [p(a_i) + p(a_j) - 2p(a_i \wedge a_j)] - [(p(a_i \vee a_j) - p(a_i \wedge a_j)] \right|}{m^2} =$$

$$= \frac{\sum_{a_i}\sum_{a_j} |p(a_i) + p(a_j) - p(a_i \wedge a_j) - p(a_i \vee a_j)|}{m^2}.$$

Still another sense of inconsistency, related to Types 2 and 3 is involved in the notion that two propositions are inconsistent if they are perceived by the actor as being contradictory. This kind of inconsistency we shall call 'incongruence,' and its lack 'congruence.' It is clear that the paradigmatic case of congruence is $a_i \leftrightarrow a_j$ and of incongruence $a_i \leftrightarrow \overline{a_j}$. Hence, we may define

$$\text{(18)} \qquad \text{Congruence}(A') \underset{\text{df}}{\equiv} d_{h_1}(A, A) = \frac{\sum_{a_i}\sum_{a_j} p \oplus (a_i, a_j)}{m(m-1)}, \quad a_i \neq a_j .$$

It should also be clear that if (sets of) propositions are either highly congruent *or* highly incongruent, they will be highly systemically related, that is highly connected.[10]

We may utilise the notion of congruence to define the polarization of belief system A'. We may partition the m propositional elements of our belief systems into k (nonempty) subsets $(1 \leqslant k \leqslant m)$ in $(m-1)$ ways. Consider the partition into two nonempty subsets, denoted $A_1^{(h_1)}$ and $A_2^{(h_1)}$, such that

$$\text{(19)} \qquad \sum_{A_1}\sum_{A_1} d_{h_1}(A_1^{(h_1)}, A_1^{(h_1)}) + \sum_{A_2}\sum_{A_2} d_{h_1}(A_2^{(h_1)}, A_2^{(h_1)})$$

is minimal. We shall call the set $\{A_1^{(h_1)}, A_2^{(h_1)}\}$ the minimal 2-factorization of A with respect to h_1. We shall let

$$\text{(20)} \qquad \text{Polarization } (A') \underset{\text{df}}{\equiv} d_{h_1}(A_1^{(h_1)}, A_2^{(h_1)}) .$$

A value of zero would represent complete absence of polarization; a value of one would represent complete polarization. The components $A_1^{(h_1)}$ and $A_2^{(h_1)}$ may, very loosely speaking, be regarded as the positive and negative components of a belief dimension. Our model thus offers the possibility of a clear operational test of the cognitive dissonance reduction hypothesis that a cognitive system will tend toward unidimensionality and balance. In our terminology, a belief system may be said to be perfectly balanced if and only if it may be partitioned into principal 2-components such that[11]

$$\text{(21)} \qquad d_{h_1}(A_i^{(h_i)}, A_i^{(h_1)}) = 0$$

$$\text{(22)} \qquad d_{h_1}(A_i^{(h_i)}, A_j^{(h_i)}) = 1 , \quad i \neq j .$$

So far, our discussion of the properties of our model has been rather abstract, and we have not dealt with the likely empirical utility of our model. It might appear that the level of sophistication required on the part of the actor to be able to provide the requisite probability data is absurdly high, and ipso facto restricts our possible universe of empirical applicability tremendously; since even with an 'elite' sample it is unclear whether inchoateness and instability of beliefs may make meaningful data generation impossible. (Cf. Converse, 1970). A preliminary test of our model, a report on which we are preparing (Grofman and Hyman, 1973) suggests that freshman and sophomore college students, at least, have no difficulty in providing the requisite data. Moreover, their probability estimates satisfy the consistency requirements of the model to a remarkable degree, and when they violate these requirements, they usually do so only by a few percentage points – a rather commendable result given that our test required them to be consistent to 0.01 accuracy.[12]

We see as the chief advantages of our approach its ability to provide a simple, unified, operationalizable and mathematically quite powerful way to measure (and distinguish among) such important aspects of individual belief systems as their consistency, connectedness, and polarization; its suitability for graph theoretic mapping of individual belief systems; and its potential use in the measurement of differences in beliefs. Its applicability to probability assessments on a non-percentage sort, is, however, an open question.

There are two other important difficulties in applying our model to empirical data, but both difficulties are shared by most other approaches to the study of attitudes. On the one hand, we have no clear way to initially determine the appropriate universe of propositions whose systematic properties we wish to examine; and on the other hand, we have no way to weight the elements of our belief system. The first difficulty may be coped with in large part by extensive pre-test. Also, more use might be made of free association procedures. The subject could, for example, be given certain propositions and asked to generate other propositions that he feels are related to the given one (McGuire, 1968, p. 65). The second difficulty, too, may not be insuperable and might be ameliorated by combining our technique with some scaling technique for ascertaining belief salience. Taking this approach would enable us to compare beliefs in terms of both salience and centrality. We concur however with Shepsle (1971, p. 792) that 'this theoretical void, i.e. a measure of (political) salience, needs desperately to be filled."[13]

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NOTES

* An earlier version of this paper was delivered at the 1969 American Political-Science Association, Chicago.

[1] For a discussion of 'lag time' phenomena see McGuire (1960b).

[2] For an excellent introduction to and bibliographic references on subjective probability see Kyberg and Smokler (1964).

[3] For a discussion of the distinction between quantified and unquantified variables see Mendelson (1964), Chapter II.

[4] These axioms are stronger than those sometimes assumed. See Davidson and Suppes (1957).

[5] One axiomatic formalization of these rules may be found in Nidditch (1962), pp. 7–8.

[6] $p(a_1)=p(a_2)$ if and only if $(a_1 \rightarrow a_2) \wedge (a_2 \rightarrow a_1)$, i.e. iff $a_i \leftrightarrow a_j$.

[7] We shall say that proposition a_j is independent of proposition a_j iff $p(a_i \mid a_j)= =p(a_j \mid a_i)$, or equivalently iff $p(a_i \wedge a_j)=p(a_i)p(a_j)$, or equivalently iff $p(a_i \mid a_j)= =p(a_i)$. These equivalences are demonstrated in Feller (1957).

[8] Other senses will be explicated below.

[9] We might also note that if we let

$$\alpha_{h_i}\ (a_i, a_j) \underset{\text{df}}{\equiv} 1 - d_{h_i}\ (a_i, a_j),$$

then many of the α_{h_i}'s possess the properties usually ascribed to measures of association. For example, $\alpha_{h_1}(a_i, a_j)$ is simply Kroeber's W, and $\alpha_{h_1}(a_i, a_j)$ has the well known property of being 1 if and only if $ad=bc$.

The characteristics of the various α_h 's and their one-to-one link to a truth functional relationship such as implication, equivalence or independence seems to us to be relevant to the controversy in the statistics literature over the properties which measures of associations ought to possess and argues for the (by now noncontroversial) position that different properties (and different measures) ought to be sought depending upon the hypothesis being tested. These points have been elaborated in Bernard Grofman, 'Measures of Logical Association', State University of New York at Stony Brook, 1971. For more on this issue see McGinnis (1958) and Majone and Sanday (1971).

[10] Cf. Georg Simmel's argument that negative relationships are still relationships, and not to be confused with the absence of relationships.

[11] This result follows (with appropriate change of notation) from Theorem 13.2, page 342 in Harary *et al.* (1965).

[12] Our questionnaire requested students to estimate the percentage of students at Stony Brook (Smith) who (1) had smoked pot at least once, (2) favored legalized abortion, (3) favored the legalization of marijuana, (4) thought that progress in securing jobs and housing for blacks has been too slow, (5) were Jewish. Then they were asked to estimate the percentage of students at Stony Brook (Smith) who shared two of these traits, e.g. "What percentage of Stony Brook (Smith) students do you think have smoked pot at least once and would also favor the legalization of marijuana." Note that three of the properties referenced involve affect; (*favor* legalized abortion, *favor* legalized marijuana, progress for blacks *too slow*); one involves overt behavior (*smoke pot* at least once), and one involves an attribute (*are Jewish*). This mix was deliberately used to demonstrate the potential range of applicability of the model. Students were instructed to fill out the questionnaire as precisely as they could and to think before they answered. Because of possible ambiguities in the wording of the questions

about conjoint attitudes (attributes), the experimenter(s) explained that these questions "asked for the percentage of Stony Brook (Smith) students who were both ——— and ———."

When the relationships among attitudes (attributes) are being examined, the questionnaire requires $k + \binom{k}{2}$ questions; in this case 15 ($k = 5$). In addition, students were asked whether they themselves "had smoked pot at least once," etc. These questions were added to enable us to test the hypothesis that students with given attitudes (attributes) would overestimate those attitudes (attributes) in the general student population. The questionnaire totalled 20 questions; it required, on the average, (including instructions) about 12 minutes to complete.

13 Sheplse (1971), p. 791.

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