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Group Decision Making over Multidimensional Objects of Choice

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Recent work in social choice theory has suggested very pessimistic results about the way in which majority rule voting procedures (a) reflect (or fail to reflect) the will of the majority and (b) are subject to manipulation. We briefly review this work but focus on more recent results that make use of the concept of the "yolk" (McKelvey, 1986, *American Journal of Political Science*, **30**(2), 283–315). This recent literature (e.g., Feld, Grofman, & Miller, 1989, *Mathematical and Computer Modeling* **12**(4/5), 405–416) shows that, when groups are voting over a set of objects that can be characterized as points in a multidimensional space under a simple sequential procedure of single elimination, alternatives that are centrally located within the pareto set are almost certain to be chosen. Moreover, the structure of majority rule in the spatial context imposes a natural "fuzzy ordering" on alternatives. We also look at how supramajoritarian decision rules can give rise to cores in spatial voting games.

Voting is a fundamental process for dispute resolution. There are numerous types of voting rules by which groups can choose one object from among some set of mutually exclusive alternatives (Black, 1958, Riker, 1982). This set can be either finite or infinite. These alternatives can be individual "candidates," possible party "platforms," or possible legislative bills. Under some voting procedures, all alternatives are rank ordered by the voters; in others, there is a sequence of balloting that at each step reduces the set of still viable alternatives. In most voting procedures in common use each voter has an equal weight in determining the outcome, but, for some types of rules, the class of weighted voting rules, different

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voter's votes may be weighted unequally (e.g., according to shares of preferred stock owned). Most voting outcomes are determined by either plurality or majority, but in some cases supermajorities (e.g., two-thirds or three-fourths) are required before a previously chosen alternative can be replaced. In this paper we initially deal with the most common case, unweighted voting under majority rule, and then extend our results to allow for differential weighting and for supramajoritarian decision rules, respectively.

We focus on one important type of sequential procedure, pairwise voting. This sequential process is analogous to the children's game, "King of the Hill." One alternative is initially designated as "King of the Hill." A second alternative is proposed to challenge it. The group then chooses between the two alternatives. The winner of the pairwise balloting is the new King of the Hill. On the next round of balloting a new alternative is proposed to replace the reigning King. Again a pairwise vote takes place and again either the old King of the Hill remains in place or a successor is chosen. Once an alternative has been defeated it is no longer eligible to enter subsequent rounds. The process continues until a final winner is chosen on the "last" round.

In the simplest variant, the process proceeds for a fixed number of rounds, with challengers entering the balloting in a predetermined order. If there is some individual who controls this ballot sequence, that person is referred to as an *agenda-setter.* Alternatively there may be a "partly random" process of selection of new alternatives, e.g., based on random recognition of proposals from the floor, with some rule that permits the group to close-off voting. In one important common variant, there is a predesignated alternative (the "status quo" or "no bill") that is always the challenger on the last vote.

One useful way of thinking about voter choice is as a choice among objects that can be characterized as vectors of salient attributes. If these attributes can be thought of as points on a continuum (especially if they can be thought of as proposed positions on a policy dimension), then we can regard voters as choosing among alternatives that are represented as points in a multidimensional space.

We now turn to a review of some of the recent literature on majoritarian voting processes, after which we provide an intuitive geometric approach to understanding group decision making where alternatives are embedded in a multidimensional space.

LITERATURE REVIEW

Consider a set of objects (alternatives) that can be located as points in a multidimensional (Euclidean) space. Now consider a group of voters (for simplicity, N, odd) trying to reach agreement on which one of these

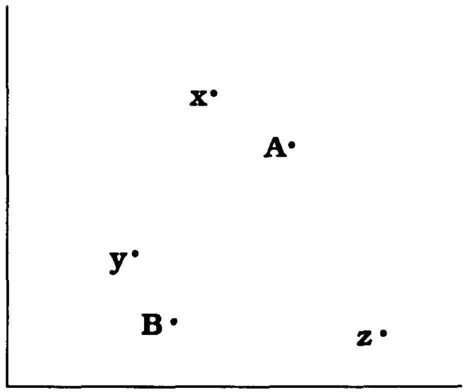

FIG. 1. Paradox of cyclical majorities.

objects will be chosen by the group. Assume that each voter has an "ideal point" in the space that is that voter's most preferred location, and that each voter prefers points closer to that point to points further away.[1] If there are three or more voters and three or more alternatives, then, if there are two or more dimensions on which the objects of choice are being located, almost never will there be a single alternative that is majority preferred to each and every other feasible alternative in the space. Such an alternative, if it exists, is known as a majority winner (a.k.a. Condorcet winner). We show in Fig. 1, for three voters and three alternatives, an example of the so-called "paradox of cyclical majorities" (Black, 1958), i.e., a situation where majority rule is intransitive. In this situation there is no majority winner.

It has been known for some time that there will be a majority winner if all voter ideal points are located in a single dimension, that is, on a straight line. In one dimension, the majority winner is the median alternative on the line, i.e., the alternative closest to the ideal point of the median voter (Black, 1958). That result can be generalized. In two dimensions, define a *median line* as one on which half or more of the voter ideal points lie above or on, and half or more of the voter ideal points lie on or below the line.[2] Then

THEOREM. *(Davis, DeGroot, & Hinich, 1972; see also McKelvey & Wendell, 1976). There exists a majority winner if and only if there exists*

[1] These assumptions give us what are commonly called Euclidean preferences, with circular indifference curves. While the results we state can be reformulated to hold for more general types of preferences, for ease of exposition we restrict ourselves to Euclidean preferences and consider only two-dimensional examples.

[2] In more than two dimensions, we replace median lines with median hyperplanes.

a voter's ideal point, M, such that every line passing through it is a median. If so, the alternative most preferred by the voter whose ideal point falls at M will be a majority winner.

A simple proof of this result is given in Feld and Grofman (1987). The result is an extremely restrictive one. It shows that only when voter ideal points are located with a kind of bilateral symmetry can we expect that there will be a majority winner. Not only will there in general be no majority winner in what is called a "spatial voting game," but, in such situations, almost anything can happen. In particular, if we let P denote the majority preference relation, then

THEOREM. (*McKelvey 1976, 1979*). *In spatial voting games, if there is no majority winner, then there exists a path between any two points r and s such that r and s are a part of a cycle; that is, there exists x, y, etc., such* $_rP_x$, $_xP_{\_}$, . . . , $_{\_}P_s$ *and similarly* $_sP_y$, $_yP_{\_}$, . . . , $_{\_}P_r$. *In other words, if there is no majority winner then every point can be made part of the top cycle.*

A simple proof of this result is in Feld and Grofman (1987). The theorem above has the disturbing implication that, in the spatial context, if there is no alternative that is majority preferred to every other alternative, then there is always some chain of alternatives that can move the group by a path of majority preference from any alternative, however popular, to any other alternative, however disliked. This shows the extraordinary potential instability of majority vote procedures and the seemingly tremendous opportunities for agenda manipulation. By voting on a finite set of alternatives in a specified order under a sequential elimination rule, it would seem that *any* alternative can be made the group choice, much like a magician "forcing" a card on a naive subject (see Riker, 1980, 1982; cf. Schofield, 1978; Bell, 1978; Cohen & Matthews, 1980).

There have been several lines of research responding to the fundamentally pessimistic results about majority rule given above. The dominant line of research has been one that looks for *structure-induced equilibria*. Such equilibria occur because of imposed limitations on the set of alternatives that can be considered, e.g., a germaneness restriction that effectively limits alternatives to a single dimension, a mandatory final vote against the status quo, a budget constraint, a closed rule that requires a yes–no vote on a single alternative proposed by an agenda setter (McCubbins & Schwartz, 1985; Riker, 1980; Romer & Rosenthal, 1978; Shepsle, 1979; Shepsle & Weingast, 1981, 1984; Feld & Grofman, 1989). Our focus in this survey, however, will be elsewhere.

In this review we focus on two other recent lines of research: those based on a geometric concept called the yolk (McKelvey, 1986) and those based on replacing simple majority rule with some supramajoritarian (greater than simple majority) requirement (Greenberg, 1979; McKelvey & Schofield, 1986, 1987; Schofeld, 1983, 1984a, 1984c, 1986b; Strnad, 1985; Schofield, Grofman, & Feld, 1988).

The *core* of a voting game is the set of undominated outcomes; i.e., those which are in place cannot be overturned. If games have a core, we expect that outcomes will be in the core and thus predictable. For a simple majority rule voting game with no ties, the core, if one exists, is simply the majority winner. But although in general there will be no majority rule winner for supramajoritarian rules there will usually be a core in a spatial voting game. Below we review results that show how much greater than simple majority is required in order to obtain a core.

The *yolk* is the minimum sphere that intersects all median lines,[3] a concept invented by Richard McKelvey (1986). The yolk can be thought of as a weakening of the core. It is of radius zero when there is a core. We present results below that show that agendas that move toward the center of the yolk are much easier to construct than agendas away from the center, and that any alternative must be majority preferred to any alternative that is 2 yolk radii further away from the center of the yolk than it is. Thus, the smaller the yolk, the greater the predictability of outcomes and the harder it is to manipulate outcomes.

Another line of recent research on choice over multidimensional issues, which, because of space restrictions, we will mention only briefly, is work on the uncovered set and various subsets thereof such as the Banks set and the Schattschneider set—sets whose location in spatial voting games can be related to the location of the yolk (Banks, 1985; Bordes, 1986; Feld, Grofman, Hartley, Kilgour, & Miller, 1987; Feld & Grofman, 1988; McKelvey, 1986; Miller, 1977, 1980, 1983; Miller, Grofman, & Feld, 1989; Moulin, 1984; Shepsle & Weingast, 1984). The *uncovered set* is the set of points that are majority preferred to all other alternatives either directly *or* at one remove; i.e., if x is uncovered, then for all y either ${}_xP_y$ or there exists z such that ${}_xP_z$ and ${}_zP_y$. The uncovered set can thus be thought of as another kind of weakening of the concept of the core. The *Banks set* is the set of maximal acyclic trajectories. It can be shown that the only possible outcomes of voting under standard (single deletion) amendment procedure when voting is sophisticated in the sense of Farquarson (1969) are those located in the Banks set. The *Schattschneider set* is the set of possible generalized medians given all possible rotations of the defining axes of the space. These sets specify the domain of feasible agenda manipulation. It can be shown that the uncovered set must be within a sphere

[3] See footnote 2.

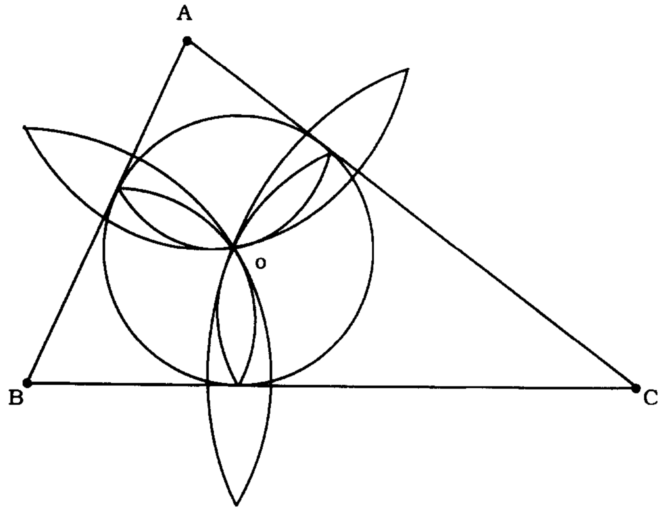

FIG. 2. The petals of the win set and half-win set around *o*, the Center of the *Y*olk.

centered around the yolk with radius 4 times that of the yolk.[4] Similarly, yolk-related bounds can be found for the location of the Schattschneider set and the Banks set.[5]

Now we turn to a presentation of the basic results, along with simplified theorem proofs.

THE YOLK AND LIMITS TO AGENDA MANIPULATION

Figure 2 shows the yolk for three-voter ideal points. It should be clear that any line passing through one ideal point and intersecting the triangle

[4] Relatedly, Packel (1981) and Ferejohn, McKelvey, and Packel (1984) have shown that, in spatial voting games, there may be *probabilistic* convergence of outcomes to a small and well-defined area of the space centered around the yolk (see also Ferejohn, Fiorina, & Packel, 1980).

[5] Three other lines of research, each of which can be thought of as providing an alternative weakening of the concept of the core distant from the center of the yolk, take us beyond the scope of this review: (1) *Von-Neumann Morgenstern externally stable solution set* of minimal area (Wuffle, Feld, Owen, & Grofman, 1989) (A V-M externally stable solution set has the property that, for any alternative outside the set, there exists an alternative in the set which beats it.); (2) the *Copeland winner,* the alternative which is defeated by the fewest other alternatives (Copeland, 1951; Glazer, Grofman, & Owen, 1985; Grofman, 1972; Grofman, Owen, Noviello, & Glazer, 1987; Henriet, 1984; Owen & Shapley, 1985; Straffin, 1980); and (3) the Borda winner (Black, 1958; Feld & Grofman, 1988).

Tight bounds on the location of the V-M externally stable solution set of minimal area can be stated relative to the center of the yolk. Similarly, the Copeland winner must lie within the uncovered set, but will not in general coincide with the center of the yolk. The Borda winner, however, can be very far from the center of the yolk.

is a median line, and the circle inscribed in the triangle is the smallest circle intersecting all median lines.

Consider the point *o* at the center of the yolk in Figure 2. The set of alternatives that are majority preferred to the center of the yolk is indicated in Fig. 2 by a "flower" pattern. It is clear that while there are alternatives that are majority preferred to *o*, they are relatively close to *o*. As we will show in the next section, the center of the yolk is the point such that the circle which encloses the *win set* of *o* (the set of points that beat it, shown in Fig. 2 as the petals around *o*) is of minimal area. We should also note that points in the yolk also have win sets whose circumscribing circles are relatively small; i.e., those points are beaten only by points relatively close to themselves.

Now consider a situation with nine voters who differ in their ideal points in a space (perhaps left–right for economic liberalism and conservatism, and down–up representing social liberalism and conservatism). Figure 3 illustrates the situation that occurs with the votes evenly spaced around a circle.

As is usual in multidimensional situations, there is no alternative in the situation shown in Fig. 3 that is majority preferred to all others. However, the yolk (the inner circle shown in Fig. 3) is a very small circle relative to the area bounded by the voter ideal points.

In Fig. 3, we show an agenda from *o* to *b*, a set of majority moves that will go from *o* (at the center) to *b* further out. As can be seen, the agenda includes three intermediating alternatives. Each pairwise choice moves

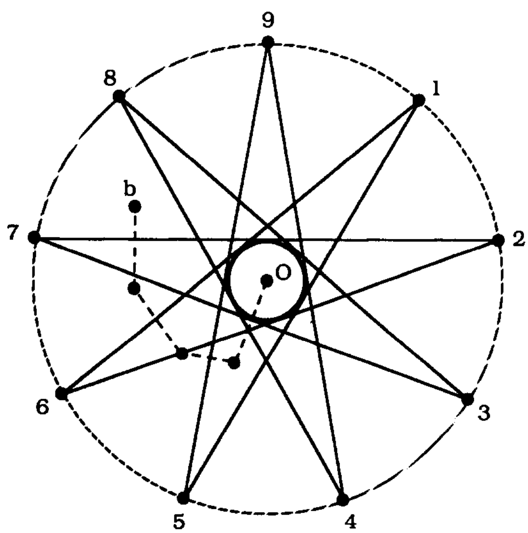

FIG. 3. Yolk for a symmetric nonagon.

the outcome slightly further out from the center (not always in the same direction). If *b* were still further away from *o* an agenda needed to make it the winning outcome would have to contain a large chain of intermediating alternatives.

DEFINITION 1. The *half-win set* of an alternative *x* is the set of alternatives that are halfway (in spatial terms) between *x* and the boundary of the win set of *x* along a ray from *x*.

THEOREM 1. *(An equivalent definition of the yolk). The yolk is the smallest circle surrounding a half-win set of a point.*

Before proving this theorem we will prove a lemma that provides further understanding of the geometry of the situation.

LEMMA 1. *An alternative, a, is preferred to an alternative b if and only if the median line perpendicular to the ab line is closer to a than it is to b.*

Proof. Every voter votes for the alternative that is closer to his/her ideal point. The division of votes is determined by the perpendicular bisector of the line *ab;* i.e., all voters on the *a* side are closer to *a* and choose *a,* and all voters on the *b* side are closer to *b* and choose *b*. If the median line is on the *a* side, then at least half the voters are on the median line or closer to *a* (by the definition of a median line), and a majority of voters choose *a*. Q.E.D.

Lemma 1 is all that is required to prove the contention in Theorem 1 that the yolk can be equivalently defined as the smallest circle surrounding a half-win set.

The half-win set of a point is illustrated in Fig. 2.

Proof of Theorem 1. From Lemma 1, an alternative in a particular direction from *o* that is closer to the median line perpendicular to that direction than is *o* is majority preferred to *o*. In other words, all alternatives between *o* and the median line and all alternatives up to and equidistant on the other side of the median line are majority preferred to *o*. This, the furthest member of the win set of *o* in a particular direction is twice the distance to the median line. Half the distance to the extreme of the win set is the median line. Thus, the half-win set extends to the median lines in every direction. The smallest circle touching all median lines, i.e., the yolk, is also the circle surrounding the half-win set of the circle's center. Q.E.D.

With three voters, as in Fig. 2, the win set of a point is the set of points that two of the three voters prefer to this point; each voter prefers alternatives within his/her circle through the point. Thus, the intersection of two circles (a petal) is the part of the win set that a particular pair of voters prefers. For this example there are three pairs of voters to be considered; thus there are three petals to the win set. For any point other than the center of the yolk, at least one of the petals is larger, and therefore the circle surrounding the half-win set is larger.

An important but simple corollary to Theorem 1 allows us to use knowledge of the size of the yolk to determine some limits on the possibilities of agendas that can be used to move the majority choice from the center of the yolk.

COROLLARY 1 TO THEOREM 1. *Given a yolk of radius r with center at o, then o is majority preferred to any point that is at least $2r$ from it.*

Proof. Since the yolk surrounds the half-win set of o, then a circle of radius $2r$ surrounds and contains the entire win set. Therefore, by the definition of the win set, there are no points outside of $2r$ from o that are majority preferred to o. Q.E.D.

This corollary indicates that an agenda manipulator cannot in one vote shift the status quo from the center of the yolk more than a distance that is specified by twice the radius of the yolk.

Theorem 1 shows limitations on the agenda manipulations that are possible if one begins the trajectory at the center of the yolk. Theorem 2 extends those results to the case of any initial starting point.

Theorem 2 specifies minimum and maximum limits on win sets of any given point with the location of the point defined in terms of its angle and its distance from the center of the yolk. The various corollaries of Theorem 2 allow us to further specify constraints on agenda manipulations. Theorem 2 can be derived from a result in McKelvey (1986).

THEOREM 2. *Maximum and minimum bounds on the size and direction of the win set of any point can be given in terms of the yolk as follows:*

I. Maximum bounds. *Let the point x be a distance d from the center of the yolk, o. Let θ be the angle between the line ox and some given vector specified direction, as shown in Fig. 4. Let r, as previously, be the radius of the yolk.*

(a) *If* $\cos\theta \leq r/d$, *then x may not be beaten by any points in the θ direction. (Note that if $\theta > (\pi/2)$, then* $\cos\theta < 0$.)

(b) *If* $\cos\theta > -r/d$, *then x may be beaten by all points within $2d$* $\cos\theta$ *$+ 2r$ along the h direction from x.*

The maximum bounds take the form of a "cardioid" (heart shape) around o with the indent of the heart at x. See Fig. 5.

II. Minimum bounds.

(a) *If* $\cos\theta \leq rd$, *then x may be beaten by all points in the θ direction.*

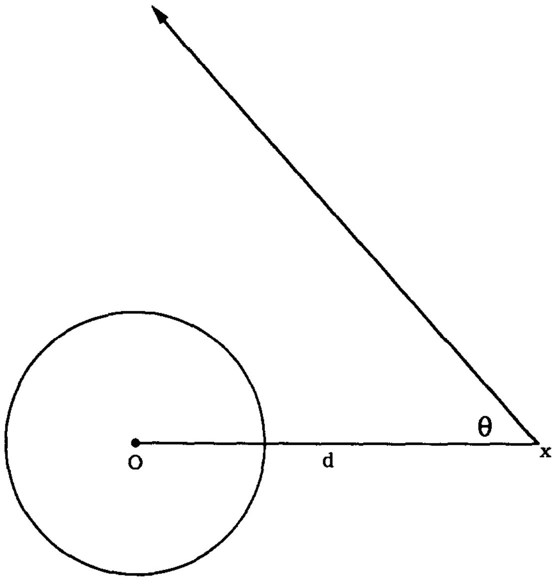

FIG. 4. Construction used in proof of Theorem 2.

(b) *If* cos θ ≥ *r/d, then x must be beaten by all points within 2d* cos θ − *2r in the* θ *direction.*

The minimum bounds take the form of a "fish" around o with the tail of the fish at x (Fig. 5). Figure 5 provides an illustration of both the maximum and the minimum win sets of a point, x.

Proof of Part I of Theorem 2 (maximum win set). Since the yolk intersects all median lines, the furthest median line from x in the θ direction is the one tangent to the yolk on the far side from x. As can be seen in Fig. 6a, this median line would be $d \cos \theta + r$ from x. Under these conditions, x would lose to all points closer to the median line than it is, i.e., all points up to $d \cos \theta + r$ on the other side of the median line from x. That is, x would lose to all points a distance $2d \cos \theta + 2r$ from x. When θ is such that $\cos \theta = r/d$, then the furthest possible median line actually goes through x itself, and x may beat all points in this direction. Figure 6b shows that if θ is larger than r/d, then the furthest median line is closer in than x, so x beats all points further out in the direction of this line. Q.E.D.

Proof of Part II of Theorem 2 (minimum win set). Since the yolk intersects all median lines, the closest median line to x in the θ direction is the one tangent to the yolk on the near side to x. As can be seen in Fig. 6a, this median line would be $d \cos \theta - 2r$ from x. When h is such that cos

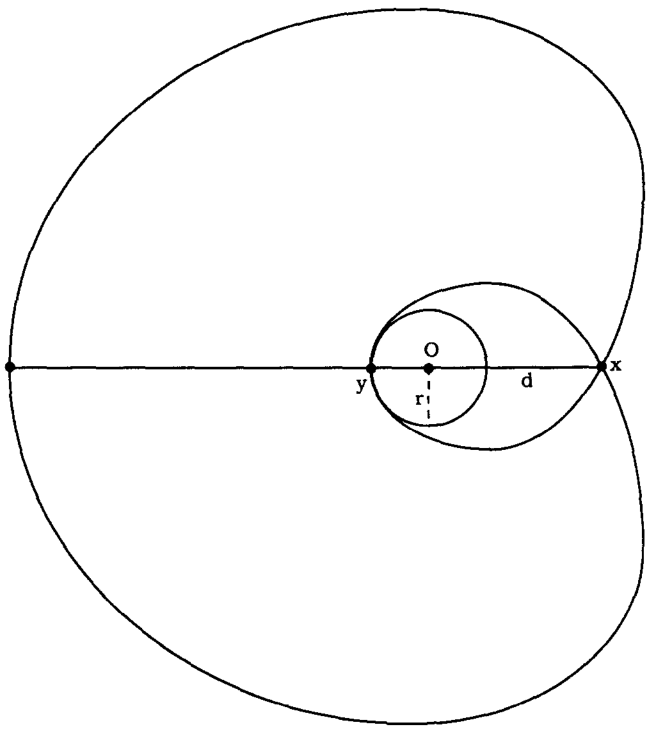

FIG. 5. Cardioid bounds on the win set of $x, d = 2.5r$ (figures only approximate).

$\theta = r/d$, then the closest possible median line actually goes through x, so x may beat all points further out in the direction of this line. Q.E.D.

Theorem 2 gives us bounds for the win set of any point expressed in terms of r. The smaller r, the smaller will be those bounds. If we have a procedure which requires a final vote against the status quo (Shepsle, 1979) then the only agenda outcomes possible lie in the win set of the status quo. The smaller r, the smaller in area will that win set be, and thus the smaller in area the set of feasible outcomes.

While Theorem 2 provides specific outer bounds of the win set of a point, some of its implications become clearer by looking at the outer bounds of the win set of a point as expressed relative to the center of the yolk. The next result is really a corollary to Theorem 2 but we have listed it as a theorem because of its special importance in understanding the limits of agenda manipulation.

THEOREM 3. *Given a yolk of radius r with center at o, then a point in the space, x, is preferred to any point in the space, y, that is further than $2r$ from the center of the yolk than is x.*

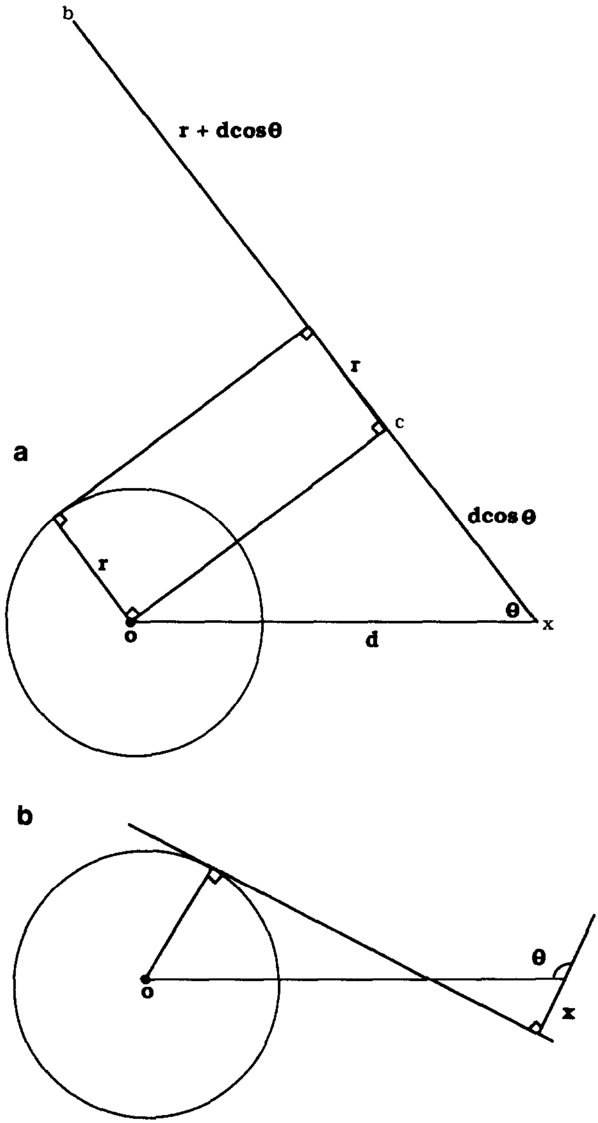

FIG. 6. (a) $\cos\theta < -r/d$. (b) $\cos\theta > -r/d$.

Proof. In Fig. 6a, it can be seen that *oc* has length 1 sin θ and *bc* has length $2r + d\cos\theta$. From the Pythagorean Theorem, it follows that

$$ob^2 = (d \sin \Theta)^2 + (2r + d\cos \Theta)^2; \tag{1}$$

i.e.,

$$ob^2 = 4r^2 + 4r\,d \cos \Theta + d^2. \tag{2}$$

Equation (2) is maximized where cos θ is maximal. This point occurs for $\theta = 0$ (at $\theta = 0$, $\cos \theta = 1$), here $ob = 2r + d$. Thus, the maximal distance that a point in the win set of x could be from the center of the yolk is $2r$ further from the center of the yolk than the point x, and this occurs directly on the opposite side of the yolk.

The smaller the size of the yolk, the more severe the limitation on the possibilities of agenda manipulation. The most severe limitation occurs in a situation where there is a simple majority winner, i.e., an alternative that is majority preferred to all others. In such a case Theorem 1 makes it clear that the yolk is just the circle of radius zero surrounding the majority winner. It follows that when $r = 0$ there is a clearly defined majority preference ordering in the entire space. Thus,

COROLLARY 1 TO THEOREM 3. *If there exists a majority winner, o, then alternative p is preferred to alternative t if and only if p is closer to o than is t; i.e., if there is a majority winner then every majority preference trajectory is acyclic.*

Proof. From Theorem 3, and the fact that the radius of the yolk is zero, it follows that any alternative is preferred to alternatives that are further away from the center of the yolk than it is. Hence proximity to the yolk defines an acyclic ordering with the center of the yolk as the majority winner.

This important but neglected result was first proved by Davis, De-Groot, and Hinich (1972; Theorem 3, 148). Note that when $r = 0$, all win sets are circular when voters have circular indifference curves. Theorem 3 allows us to specify the directionality of majority preference between any pair of alternatives. While it is generally possible for agenda manipulation to occur by tracing a path from one alternative to one other with multiple steps, a straightforward corollary of Theorem 3 provides a major constraint on the possibilities of manipulation via such multistep agendas. Q.E.D.

COROLLARY 2 TO THEOREM 3. *Let r be the radius of the yolk. If t is an alternative at a distance further from the center of the yolk than alternative p, then no majority preference path between t and p can exist with fewer than h/2r elements in it.*

Proof. Immediately follows from Theorem 3. No alternative can be majority preferred to another that is more than $2r$ closer in to the center of the yolk than itself. If the agenda is set so that each new agenda item moves the maximum possible distance of $2r$ closer in to the yolk, then it takes $h/2r$ number of steps to go from t to p.

This corollary provides a general lower limit to the length of the agenda needed to move between two alternatives. In specific situations, it will often be necessary to use even longer agendas to find a majority preference chain between two specified points, as we shall demonstrate later.

Corollary 2 to Theorem 3 can be rephrased in terms of cycles, as indicated in Corollary 3 to Theorem 3.

COROLLARY 3 TO THEOREM 3. *If p and t are two alternatives that differ by h in their distance from the center of the yolk, then a cycle containing p and t must have at least* $(h/2r) + 1$ *elements.*

Proof. Immediately follows from the previous corollary.

SPATIAL CONFIGURATIONS AND THE EXISTENCE OF A CORE FOR SUPRAMAJORITARIAN VOTING RULES

For supramajoritarian decision rules, when the dimensionality of the issue space is relatively small, recent work has shown that such frequently used rules as two-thirds and three-fourths may give rise to a set of undominated outcomes (i.e., outcomes known as a core, which once in place cannot be overturned) (Greenberg, 1979; Kramer, 1977; Strnad, 1985). Other recent work on weighted voting games (and related inegalitarian voting rules) in the spatial context has shown the sufficient conditions for there to be a core in terms of a mathematical construct called the Nakamura number (Greenberg, 1979; Nakamura, 1979; Schofield, 1984a, 1986a, 1986b). Some recent work has also dealt with the structural stability of the core. A core is said to be structurally stable if it remains in existence even if there are small perturbations in the location of voter ideal points (McKelvey & Schofield, 1986; Schofield, 1984b, 1986a).

In this section we present a powerful theorem applicable to any "proper" spatial voting game specifying the general conditions under which that rule will produce a core for any particular configuration of voter ideal points. This result is a straightforward generalization of the well-known result on majority rule spatial voting games due to Davis *et al.* (1972).

Before we restate the first result we will prove, some explicit terminology is required.

DEFINITION 2. If q is the number of votes needed to replace the status quo with a new alternative, we shall refer to the decision rule as a *q-rule*.

Let n be the number of voters. For n odd, if $q = (n + 1)/2$, we have simple majority rule. If $q > (n + 1)/2$, we have a *supramajoritarian* decision requirement.

It is often convenient to express q-rules in terms of proportions. We shall let $q^* = q/n$; i.e., q^* is the size of the minimal winning coalition expressed as a proportion q/n.

Now we provide an example of a supramajoritarian rule in which there is no core, which helps us to understand exactly what a core is in this context.

Consider the situation in Fig. 7, with all of the voters located at or outside the corners of a triangle. For a $q^* = 2/3$ rule, it should be evident that any point within the triangle can be beaten by moving toward one of the edges. Furthermore, any point that is "outside" any two of the edges, i.e., in one of the "outside corner" areas, is on the "inside" of the third edge. Therefore, every point is on the inside of one of the edges; that point can be beaten by any point closer to that inside edge. Thus, every point is beaten, and there is no core.

Figure 8 shows some median lines for a symmetric seven-voter example. (The median lines depicted are those passing through two voter ideal points; each other median line passes through only one voter ideal point.) In every direction, there exists a median line.

The idea of median lines for simple majority rule may be extended to q-rules, by introducing the idea of a "q-tube," which may be thought of, in a rough sense, as a "thick" median line.

DEFINITION 3. In two dimensions, there are supramajority q-tubes in each direction consisting of pairs of parallel lines such that, for each parallel line defining one of the edges of the tube, less than q of the points are "inside" that line (where "inside" means toward and past the other parallel line defining the tube), and less than $n - q + 1$ of the points are "outside" that line.

We show in Fig. 9 an example of several q-tubes for a nine-voter situation where q is 6. Each tube consists of a pair of parallel lines with six voter ideal points on or to one side of the line.

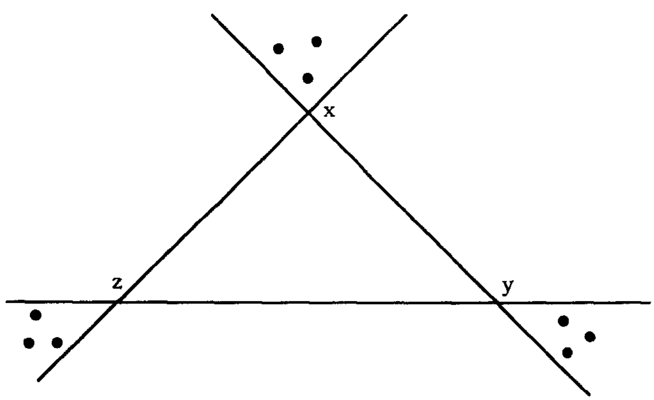

FIG. 7. A situation where $q^* = 2/3$ has no core.

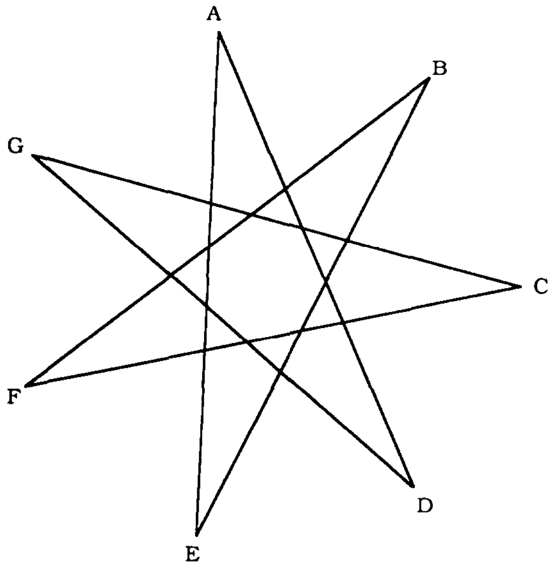

FIG. 8. Median lines in a seven-voter spatial game.

For example, to find the location of a boundary line in a given direction we simply move a line oriented in that direction perpendicularly until it cuts off exactly six voter ideal points (i.e., there are less than six voter ideal points on one side of the line, and less than four on the other).

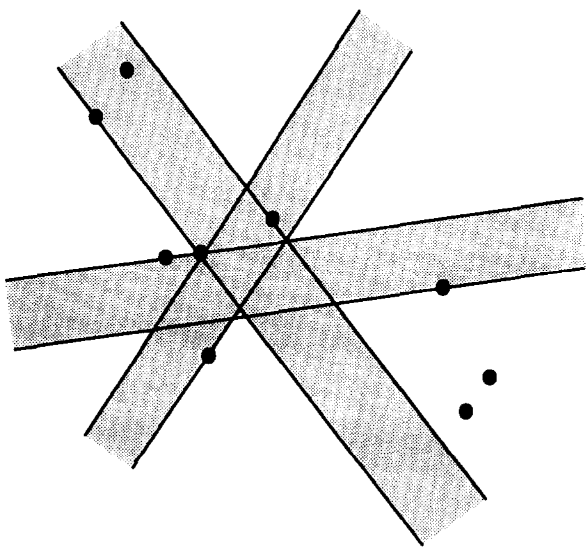

FIG. 9. Example of $q^* = 6/9$ "tubes."

For any spatial voting game in which it is possible to specify the set of minimal winning coalitions, the generalization of a median line is an *extremal* boundary tube. Before defining extremal boundary tubes, two definitions are necessary.

DEFINITION 4. In two dimensions, a *boundary line for a minimal winning coalition* is a line such that one or more voters in the coalition have ideal points on the line, and all of the other voters in the coalition have their ideal points on the same side of the line. The "loss side" of a boundary line is the side of the line that contains no voter ideal points from the winning coalition. The "win side" (including the line itself) is the side of the line that contains the entire coalition.

It should be apparent that every coalition has two boundary lines in every direction, one each on "opposite" sides of the coalition, forming boundary "tubes."

DEFINITION 5. Consider the set of all minimal coalitions corresponding to any given decision rule. The *extremal boundary line* in a direction is the boundary line in that direction that maximizes the number of voter ideal points on the "loss side" of the line.

Figure 10 provides an example. Assume that there are five voters (A, B, C, D, E) with a decision rule requiring a winning coalition to contain D and any other two voters, i.e., ABD, ACD, ADE, BCD, BDE, or CDE. Consider the vertical direction (i.e., with horizontal boundary lines). Each minimal winning coalition is bounded by an upper and lower boundary line (e.g., BCD is bounded by line 1 through B and by line 3 through

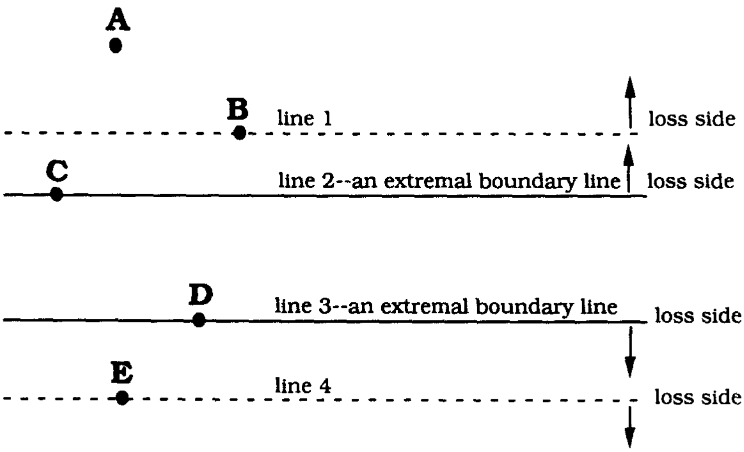

FIG. 10. Extremal boundary lines for a rule requiring D and two other voters.

D; CDE is bounded by line 2 through C and line 4 through E). When all minimal winning coalitions are examined, it can be seen that line 2 bounding the CDE coalition is the extremal boundary line maximizing the loss side at the top, and that line 3 bounding the BCD (and also the ACD and ABD) coalition is the extremal boundary line maximizing the loss side at the bottom.

As this example shows, there are two parallel extremal boundary lines at each angle in the plane (i.e., one maximizing the loss side on the top, and the other maximizing the loss side on the bottom). These two lines can be used to define an extremal boundary tube as follows.

DEFINITION 6. *An extremal boundary tube* is the set of points in the intersection of the win sides of two parallel extremal boundary lines. (In the "proper" games that we are considering, extremal boundary tubes are never empty and always include the area between the two extremal boundary lines.) In Fig. 10, the extremal boundary tube for the vertical direction is the horizontal tube defined by the area between the extremal boundary lines 2 and 3. In like manner, we can specify extremal boundary lines and tubes in any direction. Note that the "inside" of an extremal boundary tube is the intersection of the "win" sides of the two extremal boundary lines. It should also be apparent that q-tubes are just a special case of extremal boundary tubes.

Now we are in the position to provide a simple proof of our next theorem.

THEOREM 4. *(extension of theorem on majority rule in Davis et al., 1972). There is a core if and only if the intersection of all extremal boundary tubes is non-empty. If there is a core, it is that intersection.*

Proof. Any point on the loss side of an extremal boundary line is beaten by some other points, and so cannot be in a core. A point outside of any extremal boundary tube is on the loss side of some boundary line, and so is not in the core.

If a point is inside an extremal boundary tube, it is not on the loss side of any boundary line in that direction, and so is unbeaten in that direction. If a point is inside all extremal boundary tubes, then it is unbeaten in every direction and so is in the core. Q.E.D.

The above generalization of the Davis *et al.* (1972) results to all proper simple spatial games is unique to the present authors, since it requires the definition of extremal boundary tubes as the natural generalization of median lines in the simple majority case; however, a similar result can be found in Slutsky (1979), expressed in terms of gradient vectors. The mathematics underlying that proof is both more general and considerably more complex than the simple proof given above for the case of Euclidean preferences.

CONDITIONS FOR A CORE IN SUPRAMAJORITATIAN GAMES

Schofield *et al.* (1988) provide a relatively nontechnical summary of the results of work on the stability properties of supramajoritarian rules and weighted voting rules. However, the proofs of the basic theorems have been presented only very technically in specialized mathematical economics journals. We provide simplified proofs for Euclidean preferences in two dimensions for two of the most important results to date on conditions for a core in supramajoritarian and weighted spatial voting games. Our proofs are based on the concept of extremal boundary lines: these proofs should provide an intuition for the meaning, applicability, and significance of the idea of boundary tubes.

DEFINITION 7. (Nakamura, 1979). The Nakamura number, which we shall denote NN, is the smallest number of minimal winning coalitions whose intersection is empty.

To find the Nakamura number we can simply count how many coalitions there are in each set of coalitions whose intersection is empty. The smallest number of minimal winning coalitions that can have an empty intersection is NN. The Nakamura number plays an important role in understanding the behavior of weighted voting rule games (see below). Also, the Nakamura number of voting games determines the length of the shortest voting cycle permitted by the voting rule; e.g., if NN = 4 the shortest cycle permitted by the voting rules includes exactly four alternatives. (Note that whether such a voting cycle is actually found depends upon the coincidence of preferences—the fact that a majority cycle of four alternatives is permitted by the rules does not necessarily mean that four alternatives can be found that will be supported by the succession of winning coalitions that would be required for the voting cycle.)

A lemma is required for proving Theorem 5.

LEMMA 1. *The Nakamura number, NN, is the minimum number of minimum winning coalitions that need to be considered so that everybody loses at least once, i.e., such that every voter is excluded from at least one minimal winning coalition in the set.*

Proof. A set of minimal winning coalitions whose intersection is empty is also a set of minimal winning coalitions, the union of whose complements is the entire electorate, and conversely. Q.E.D.

THEOREM 5. (*Schofield, 1984a; see also Nakamura, 1979*). *In W dimen-*

sions, any game with Nakamura number greater than W + 1 must have a core.

Proof of Theorem 5. We first prove the result for two dimensions. Assume that there is no core. There is a smallest circle such that it touches all extremal boundary tubes. If the smallest such circle were a point, then that point would be a core. Since the smallest such circle is nonempty, there must be three extremal boundary tubes that are tangent to the circle; otherwise the circle could be smaller (see Fig. 11).

The three tangent boundary lines form a triangle. It can be seen that there can be no voter on the "win" side of all three boundary lines. Therefore, the intersection among the three minimal winning coalitions bounded by these boundary lines is empty. This says there are three minimal winning coalitions whose intersection is empty; so NN $<$ 3. If no core implies NN $<$ 3, then NN $>$ 3 implies that there is a core.

The proof for three dimensions is essentially the same, except that the smallest circle is a sphere, and the "triangle" becomes a pyramid. The result is proved by showing four minimal winning coalitions whose intersection is empty. The proof can be extended in this fashion for other numbers of dimensions. Q.E.D.

It is easily seen that an alternative statement of Theorem 5 is: "A game with Nakamura number, NN, must have a core in less than NN − 1 dimensions." Since the Nakamura number for simple majority rule is 3, majority rule is guaranteed to have a core in one dimension.

The Nakamura number has been defined on the basis of the rules of the game, i.e., the coalitions that are winning, without taking account of the fact that only a subset of the winning coalitions may actually form in a

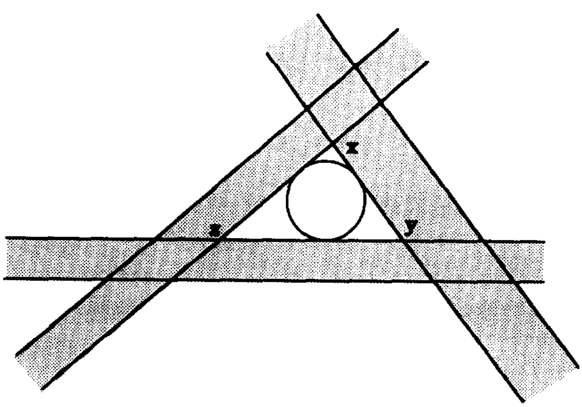

FIG. 11. The smallest circle touching all extremal boundary tubes and the three extremal boundary tubes tangent to it.

particular (spatial) context, due to other constraints: some actors may have sufficient antipathy to never join together, others may always vote together, etc. In a spatial context, we suggest that the only minimal coalitions that are feasible are those that are minimally connected, as defined below.

DEFINITION 8. (Grofman, 1982). A connected coalition is a coalition that includes all voters contained within its convex hull.

DEFINITION 9. A minimally connected winning coalition is a connected coalition that is winning, such that excluding any actor on its convex hull would make it nonwinning.

Of course, any minimally connected winning coalition must include a minimal winning coalition.

Now we can define a spatial Nakamura number in terms of the feasible coalitions in a spatial context, i.e., the connected coalitions for a particular configuration of voters.

DEFINITION 10. The spatial Nakamura number (SN) is the minimal number of minimally connected winning coalitions whose convex hull intersection is empty.

LEMMA 3. *The spatial Nakamura number can be no less than the Nakamura number, i.e., $SN > NN$.*

Proof. If there are SN minimal connected winning coalitions whose convex hull intersection is empty, then they contain SN minimal coalitions whose intersection is empty. Q.E.D.

THEOREM 6. (*Greenberg, 1979*). *In two dimensions, for q^* greater than $2/3$, there must always be a core.*

Theorem 6 is actually a special case of Theorem 5. This can be seen by using the following Lemma.

LEMMA 4. (*Schofield 1984b; see also Nakamura, 1979*). *For q-rule games, the Nakamura number, NN, is the lowest integer bound of $n/(n - q)$.*

Proof. If q of n votes are needed to win, each minimal winning coalition will exclude exactly $n - q$ voters. Only when $NN[n/(n - q)] > n$ will the set of excluded voters include the entire electorate, but this is equivalent algebraically to $NN > n/(n - q)$. Q.E.D.

Theorem 5 then follows as a corollary of Theorem 6 and Lemma 3. From Lemma 3, if $q^* > q/n > 2/3$ then $NN > 3$. From Theorem 6, there is always a core in two dimensions. The more general corollary for W dimensions can be stated as follows:

COROLLARY to THEOREM 6. *For W dimensions there is a core for any q-rule with $q > w/w + 1$.*

The Nakamura number result of Theorem 5 also applies to weighted voting games (games that deviate from the one person-one vote principle) and compound games (games that involve decisions by two or more groups).

DEFINITION 11. In a *weighted voting rule game* each player, i, has a weight p_i, and there is a q-rule, such that a motion passes if and only if the weights of the players who support it is at least q. (This is equivalent to a q^* rule with p_i values normalized so that $\Sigma p_i = 1$.)

Weighted voting games with sufficiently high supramajoritarian requirements inevitably have a core. A weighted voting game is analytically equivalent to an unweighted voting game with several voters occupying the same position., Therefore it follows from Theorem 4 that any weighted voting game with $q^* > 2/3$ has a core in two dimensions. Games with $q^* > 2/3$ that are equivalent to games with $q^* > 2/3$ must also have a core. For example, consider a game with the following weights: (.2, .2, .2, .2, .05, .05, .05) and $q^* = .65$. By inspection, this game can be seen to be equivalent to a game with the following weights: (.24, .24, .24, .24, .01, .01, .01, .01) and $q^* = .73$. The latter game has $q^* > 2/3$, so it must have a core in two dimensions; so the former equivalent game must also have a core in two dimensions. We can also get this result by ascertaining the fact that the Nakamura number of this game is 4.

DEFINITION 12. A game in which there are one or more voters who are members of all winning coalitions is said to be a veto game; players who are in all winning coalitions are said to be veto players.

In a veto game the Nakamura number is infinite, because even the intersections of the largest sets of winning coalitions are non-empty (i.e., containing the veto player). It follows that every game with a veto player has a core (Nakamura, 1979; see also Schofield, 1984a).

DISCUSSION

The results given above, which focus on the importance of the size of the yolk as a constraint on agenda manipulation and predictability of outcomes in spatial voting games, lead to a very different picture of the possibility of democratic group decision making than that in other recent pessimistic treatments such as that of Riker (1982). We believe that the existence of a "fine-structure" to collective preference in the spatial context, when combined with other formal and informal aspects of the usual types of collective decision making in small groups (including a predilection for more than bare majority agreement), makes likely outcomes that are central with respect to the set of voter ideal points.

For someone interested in organizational dynamics, especially in the processes of formal group decision making, the results given in this paper have a number of useful implications.

First, the results stated above suggest that, at least for democratic decision-making bodies using a simple sequential voting process, the distribution of voter preferences critically conditions the set of feasible outcomes. In particular, ceteris paribus, alternatives that are relatively close to the center of the yolk are more likely to be chosen by the group than alternatives that are far away. While there is a considerable body of important literature demonstrating how manipulation of the sequencing of alternatives can dramatically affect which outcome is chosen (see the excellent review in Riker, 1982), that literature hinges (a) upon voting rules that are more complex than the pairwise process used above and/or (b) upon relatively lengthy agenda sequences.

Second, the results given above provide the intuitive underpinnings for an understanding of "agenda trajectories" and the nature of the majority preference relationship in the context of multidimensional alternatives. Once we locate the center of the yolk, we know roughly what alternatives can be expected to have reasonable probabilities of success simply by comparing each alternative's relative proximity to the center of the yolk. The yolk is a critical concept in understanding the dynamics of majority rule, and extensions of that concept apply to supramajoritarian decision making as well.

Third, the above results show how the number of issue dimensions affects the magnitude of the supramajority needed to guarantee that there will be a set of undefeatable alternatives. For a given number of dimensions, for large enough supramajorities there may be a substantial zone of the space which, once entered, cannot be left; i.e., no alternative in the zone is defeated by any alternative outside the zone by the requisite special majority. Thus, supramajoritarian decision making can be a powerful force in preserving the status quo.

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