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A Review of Macro Election Systems*)

Acknowledgements

My interest in this area was first stimulated almost a decade ago by Duncan Black's classic work, *The Theory of Committees and Elections.* A major source for this review is Doug Rae's *The Political Consequences of Electoral Laws* and the work done by Rae and members of the University of Essex. I'm indebted to Steven Brams for making available to me a prepublication copy of his book, *Game and Politics*, and to William Lucas for providing me a copy of his review monograph on weighted voting. From these two works my treatment of cumulative voting and weighted voting has largely been adapted. To the members of the 1975 ECPR "Workshop on Parties" my thanks for a stimulating discussion and a number of helpful suggestions for revisions in my paper, with special thanks to Professor Giovanni Sartori, who has usefully called to my attention the importance of variables such as district size, party cohesiveness, and transferability of voter allegiances to an understanding of the long-run dynamics of party proliferation and decay. Finally, I'd like to acknowledge my deep indebtedness to Rudolf Wildenmann for having made possible my appointment at the University of Mannheim and the collaborative crosscultural research which has come about as a result of it.

I. Introduction

Our aim in this paper is the quite limited one of surveying recent work on mathematical models of macro election systems. By an *election system (voting scheme)* we shall mean a scheme for expressing voter preferences in terms of a ballot (or ballots) and for transforming the set of voter

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preferences so obtained into an election outcome[1]). (See Figures 1 and 2.) We shall *not* deal here with other aspects of electoral systems such as ballot-format[2]), candidacy and suffrage requirements[3]), or gerrymandering[4]); nor shall we discuss forms of representation based on groupings other than territorial ones[5]); nor shall we attempt a review of the vast number of empirical studies on parties and elections. We shall focus our attention on:

a) the expected similarity of outcomes under various voting schemes;
b) the relationship between seats and votes for various election systems — in particular the conjectured cube-law relationship between seats and votes in two-party single member district plurality elections, and thresholds of exclusion and representation in multiparty contests as a function of district size and of distribution of party strength across election districts;
c) balance of power and strategic considerations in weighted voting and cumulative voting systems;

It is not our aim in this paper to try to rehash (much less resolve) the normative and empirical issues concerning the desirability of alternative voting schemes, but we shall provide some discussion of some of the ways in which recent formal work has shed new light on some rather old questions. We shall confine our attention to those voting schemes (of the ones enumerated in Figures 1 and 2) which have been used for legislative elections. Thus, we shall not try to deal with voting procedures internal to legislatures or other small groups. The reader interested in formal

[1]) We shall use the terms election system and voting scheme interchangeably throughout.
[2]) E. g. Campbell (1962) has found a relationship for the U.S. between the type of ballot and the incidence of straight party voting. "We find, in the states which make it relatively easy for the voter to mark a straight ticket, that the number of straight tickets marked is some 20 percent higher than in those states where the ballot requires a series of separate decisions among the candidates for each of the various offices."
[3]) E. g. a number of U.S. studies (Shinn 1971; Kelley, et. al, 1967) have shown that the type of registration procedures and length of residency requirements may have dramatic effects on voter registration and turnout. On candidacy and suffrage requirements generally see Ross (1955).
[4]) Useful studies of U.S. constituency engineering practices may be found in Polsby (1971).
[5]) E. g. "Occupational representation was offered after the First World War as the panacea for the admitted shortcomings of territorial representation." (Friedrich, 1968, p. 282.) Grouping by ethnicity on separate electoral rolls has also been a common practice in many countries (See Laponce, 1957, pp. 321—325). In the U.S. there has recently been revived interest in imposing quotas for the representation of racial, age, and other groupings thought to be underrepresented in the political system (See, eg. Cavala, 1974).
N. B. In some cases the territorial aggregation which forms the constituency grouping may be entire nation-state, eg. Israel.

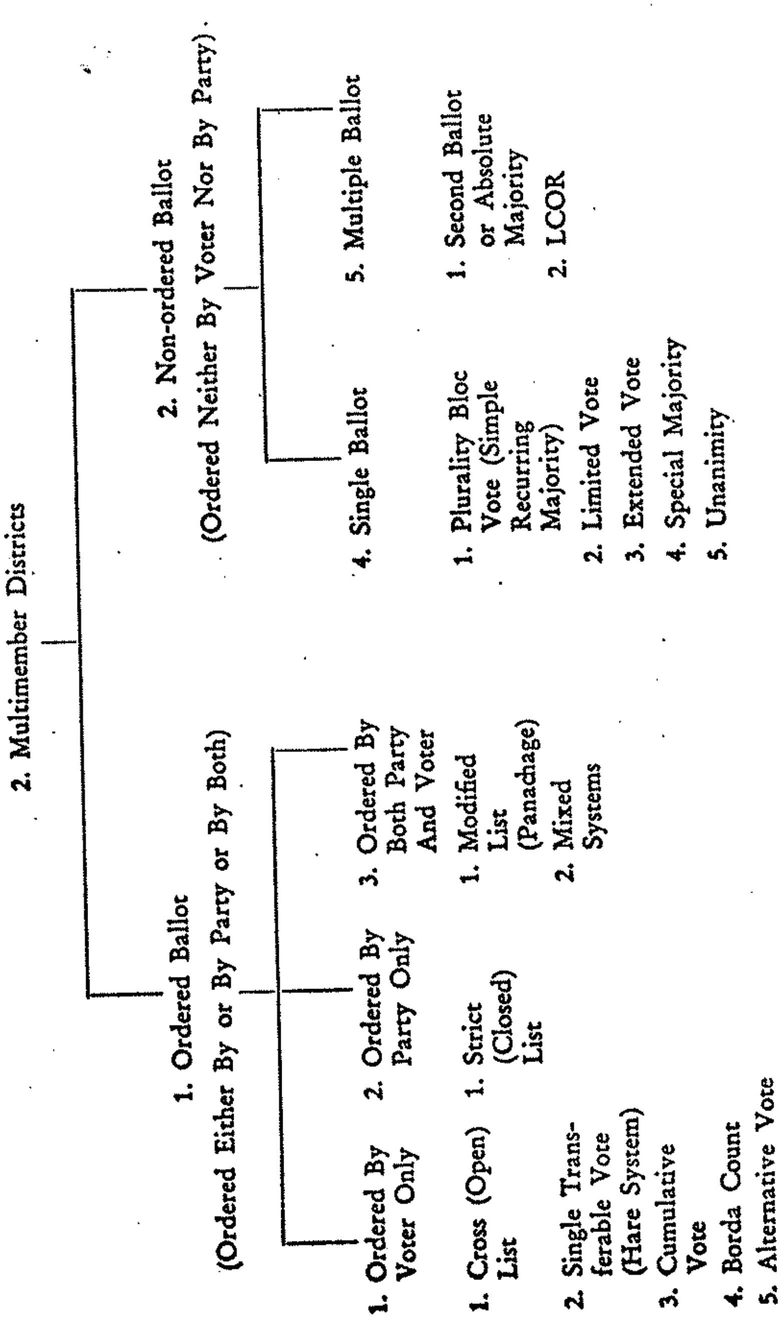
Figure 1

Types of Election Systems — Single Member Districts — Single Tier

- 1. Single Member Districts
  - 1. Ordered Ballot (Ordered Either By Voter or By Party or By Both)
    - 1. Ordered By Voter Only
      - 1. Borda Count, smd
      - 2. Alternative Vote, smd (Also Grand Junction System, Manson System, Hallett System)
    - 2. Ordered By Party Only
      - None
    - 3. Ordered By Both Party And Voter
      - None
  - 2. Non-ordered Ballot (Ordered Neither By Voter Nor By Party)
    - 4. Single Ballot
      - 1. Plurality, smd
      - 2. Special Majority
      - 3. Unanimity
    - 5. Multiple Ballot
      - 1. Second Ballot or Absolute Majority, smd
      - 2. LCOR, smd
      - 3. SAP
      - 4. FIB

Figure 2

Types of Election Systems — Multimember Districts — Single Tier



research into this area might, for an introduction to the voluminous recent literature, consult Black, 1958; Buchanan and Tullock, 1962; Murakami, 1968; Grofman, 1969; Farquharson, 1970; Niemi and Weisberg, 1972; Sen, 1971; Pattanaik, 1971; Riker and Ordeshook, 1973; Fishburn, 1971, 1973, 1974c; Shepsle, 1973; Grofman, 1975b; and the references cited therein.

II. Typologies of Election Systems

Proportional representation has been a subject of heated controversy for over a hundred years. Adoption of PR has been held by many of its opponents to lead inevitably to factionalism and immobilise, while its advocates have held it to be the only means to insure that all segments of the electorate are fairly represented[6]). Unfortunately much of the debate over PR (especially the early debate) is marred by a failure to distinguish between the principle of PR and its actualization in a number of quite different voting procedures, each of which can be regarded as providing some extent of minority representation but which differ considerably in their probable consequences for the representation of minorities and for the cohesiveness and stability of (and patterns of competition between) parties. Indeed, the term proportional representation system is misleading and there is disagreement in the literature as to which systems the label may rightfully be attached (e. g., schemes like the cumulative vote and the limited vote are sometimes treated as proportional and sometimes are put in an in-between category [see Lakeman, 1974, p. 176]). J. F. S. Ross proposed to reserve the term proportional representation not for any particular set of voting schemes, but rather for the *principle* that the distribution of seats among parties should correspond with the distribution of votes among them.

> "(A)11 election systems show some sort of relationship between the number of votes cast for a party and the number of seats that that party obtains: that is, they all give *some* degree of proportional representation ... In every case the party with a great many supporters gets a larger measure of success than does the party with comparatively few. Where the systems differ in this respect is just in the extent and the reliability of the provision they make for the correspondence between seats and votes." (Ross, 1959, p. 59.)

We shall follow Ross's advice and rather than talking about PR and non-PR systems we shall distinguish between majoritarian and non-majoritarian systems (where that distinction can be made quite precise) and then seek to evaluate for each of the various voting schemes (whether majoritarian or non-majoritarian) the conditions under which particular relationships result between percentage seats obtained and percentage votes obtained.

Of the election systems listed in Figure 2, the various forms of party list systems, the single transferable vote, the cumulative vote, the Borda count and the limited vote are all non-majoritarian schemes, where by a *majoritarian scheme* we simply mean a voting scheme which has the property that there always exists some coalition consisting of only a bare majority of the voters which can be assured of obtaining *all* of the seats if it coordinates members' ballot choices[7]).

Within majoritarian schemes, schemes requiring absolute majorities (e. g., alternative vote) may be distinguished from those requiring only relative majorities (e. g., first-past-the-post simple plurality). With list systems the most important distinctions are between the various forms of quotas and quotients[8]) (see Figure 3).

[6]) "(I)f the assembly becomes divided into so many factions that there cannot be found a stable majority for the executive's support, all government activity becomes paralyzed. Impressive documentation has been assembled to show how in country after country precisely this paralysis has crept in" (Friedrich, 1968, p. 295). In the Republic of Ireland which has used the Hare System since its creation, the largest party, the Fianna Fáil has in the last two decades sought to switch to a first-past-the-post system. In national referenda in 1959 and 1968 the Hare System, first narrowly, then by a 3 to 2 margin, was endorsed by Irish voters. It's instructive to quote from the arguments of one Fianna Fáil deputy (Booth, 1959, pp. 10—11).
"The main weakness and danger of a proportional electoral system has been excellently summarized by Deputy J. A. Costello as being one that leads to a large number of small parties being returned, making for instability in government. That is inherent in the system of P. R. and the single transferable vote ...
The basic fallacy in the argument favoring proportionalism is that 'representation' is assumed to be the test of government; in fact, the main test is a government's capacity to govern ... While a strong government is not necessarily a good or a wise government ... a coalition by its very nature must be weak and (except in wartime) cannot govern decisively."

[7]) The distinction between majoritarian and non-majoritarian schemes is inspired by the nations of minimal winning coalition and majority voting game in game theory (See Luce and Raiffa, 1957).

[8]) Within list systems, attention is also paid to the availability of voter options to affect the ordering of candidates ordained by the parties and/or to vote for candidates of more than one party. This distinction we have incorporated into our own typological scheme (See Figure 2).

Figure 3

Quotients and Quotas for Various Types of List PR Systems

1. D'Hondt Highest Average Formula	quotients+ = $\frac{\text{total votes for party } i}{1, 2, 3, 4, \ldots, m}$
2. St. Lague Highest Average Formula	quotients = $\frac{\text{total votes for party } i}{1, 3, 5, 7, \ldots, 2m-1}$
3. Greatest Remainder Formula	quota for party i = $\frac{\text{total votes}}{m}$
4. Modified St. Lague Highest Average Formula ‡	quotients = $\frac{\text{total votes for party } i}{1.4, 3, 5, 7, \ldots, 2m-1}$
5. Imperial, Greatest Remainder Formula	quota for party i = $\frac{\text{total votes}}{m+2}$

+ m = total seats (in a given district)
‡ This is what Rae refers to as the "Lague formula".

A typology of election systems which cuts across the PR vs. non-PR (or majoritarian vs. non-majoritarian) breakdown has been proposed by Douglas Rae (1967, 1971). Rae distinguishes between ordinal and categorical ballots.

Categorical ballots, as Rae defines them, are those which compel the voter to vote for all the representatives of a party if he is to vote for any of them. Thus all single member districts (smd) non-ordered ballots are necessarily categorical because there is only one candidate of each party — hence, in voting for *a* candidate of a party a voter must necessarily vote for *all* (i. e., *the)* candidates of that party. Of course, strict party list systems are categorical. Rae, however, also (not unreasonably) lumps together all current modified party list systems as categorical on the grounds that they do not allow voters to cross party lines and, for all practical purposes, the voter who wishes to vote for any candidate of a party is, in effect, forced to give his whole mandate to that one party.

Ordinal voting schemes are, on the other hand, defined by Rae as those which allow the voter to express a more complex preference by some form of rank-ordering the parties and/or their candidates.

The typology we propose in Figures 1 and 2 involves a four part classification system[9, 10]). It distinguishes between *single-member* and *multi-member* districts but not between majoritarian and non-majoritarian schemes[11]). Instead it distinguishes between ordered and non-ordered ballots, a distinction very similar to Rae's distinction between ordinal and categorical ballots, but allowing us to make certain additional distinctions. Thus within ordered ballots we distinguish according to whether the ballot is ordered by voter only, party only, or by both, and within non-ordered ballots we distinguish between *single ballot* and *multiple ballot* procedures. Our third cross-cutting classification type is the balloting and tabulating procedure itself, e. g., non-ordered ballots in single member districts may be used in four quite different balloting schemes (the second ballot runoff scheme used in France since 1958; the low man out scheme (LCOR) common in U. S. private organizations; filling in the blanks, a rather esoteric voting scheme described in *Robert's Rules of Order* (see Grofman, 1969); and standard amendment procedure, the sequential pairwise elimination procedure used in the U. S. and Britain for dealing with mutually exclusive amendments to a main motion). Finally, we shall draw a fourth distinction between *single-tiered* and *multi-tiered* systems but we shall not deal with other than single-tiered systems in this paper[12]).

[9]) A potentially useful distinction inspired by game theory which we shall not make use of in this paper is that between binary and non-binary voting schemes. For definitions of these terms and an important but opaque treatment of the vulnerability of various voting schemes (especially binary ones) to manipulation of outcomes via deliberate distortions of the voters true preference and/or manipulation of the order in which alternatives are posed see Farquharson (1969). Treatment of Farquharson's work would require too lengthy and unduly technical a digression for our present purposes. An excellent simplified treatment of his main results is available in Brams (1975).

[10]) Clearly, however, any scheme which seeks equity between seats and votes must either involve multi-member districts and/or some form of tiering procedure. Of course, non-majoritarian schemes may also impose requirements for special majorities or unanimity such that any election will require compromise with some minority members, even though the minority is "officially" denied representation.

[11]) The classificatory schema of Figures 2 and 3 is similar, we recently learned to that proposed by J. F. S. Ross (1959, pp. 60—61).

[12]) By multi-tiered systems we mean ones such that vote residues in some constituencies are "carried over" to determine representation in another "higher level" constituency. An example of a multi-tiered system would be that of Denmark. Our distinction is not the same as Rae's (1967, 1971) distinction between "simple" and "complex" election systems. For example, the system in recent use in Germany which has voters voting in two different constitutencies only one of which is a smd is "complex" but not multitiered.
We shall also avoid dealing with complex systems in this paper. The properties of a complex system can be approximated by looking at the behaviour of its component parts. Recently in the U.S. there has been renewed interest in complex multi-stage forms of election procedures for elections such as those for delegates to the Democratic National Concention since they permit at least the appearance of greater grass-roots participation. Some formal work has recently been done on such systems but we shall not attempt to summarize it here (See Buchanan and Tullock, 1962; Miller, 1971).

III. Votes into Seats

Much of the discussion about the vote-seat relationship for various election systems has made use of rather limited data, often selected so as to make a polemical point (see, e. g., Lakeman, 1974). The first (and as far as I know, only) comprehensive comparative study is Rae (1967, 1971).

The vote-seat relationship may be approached both as a descriptive and analytic problem. For purposes of analysis of the theoretical properties of election systems three indices and one graphical technique have recently been proposed. The indices are the index of maximum distortion (Loosemore and Hanby, 1971), the threshold of representation (Rokkan, 1968; Rae, Hanby and Loosemore, 1971) and the threshold of exclusion (Rae, Hanby, and Loosemore, 1971), the index of nonrepresentation (Black, 1967); the graphical technique is the (maximum/minimum) seat/vote curve (Dahl, 1956; Grofman, 1975). To each of these analytic expressions for the theoretical limiting cases there corresponds on the one hand (in principle, at least) observed values or relationships derived from actual data on systems-in-use, and on the other hand, theoretical expected values based on assumptions about the overall party strengths and their probability distributions across constituencies.

The *threshold of representation* is the minimum support necessary to earn a party its first parliamentary seat.

> „This analysis is positive, for it attends to the conditions of success. And it is optimistic (Panglossian), for it presumes that established parties are obliging enough not to form alliances against an emergent party and even go so far as to divide their votes to its best advantage . . . But it is a telling fact that parties can easily obtain the vote prescribed by these threshold functions yet fail, in fact, to obtain representation." (Rae, Hanby, and Loosemore, 1971, pp. 479—480.)

The *threshold of exclusion*, on the other hand, is the maximum support which can be attained by a party which, nevertheless, fails to win even one seat. The threshold of representation provides a necessary condition for parliamentary representation, the threshold of exclusion provides a sufficient condition for it[13]). The analysis of the threshold of exclusion is simplified by the fact

[13]) In the analysis that follows, we shall neglect exclusion rules such as those which deny representation to parties with less than a minimum percentage of the national vote.

„(A) small party's opponents have no better strategy than either to a) let one of their number stand alone against the party in each district, or b) form a wholesale electoral alliance to oppose it in each district ... Thus, we are to suppose that ... our party with vote share V_i faces a single adversary with a vote proportion of $1 - V_i$." (Rae, Loosemore, and Hanby, 1971, p. 480.)

Let
m = number of members being elected from a given district
n = number of parties contesting the election in some given district
r = number of seats in the legislature
T_R = threshold of representation
T_E = threshold of exclusion
v_i = vote share for party i in a given district
V_i = total national vote share for party i

We shall not reproduce the anlysis of Rae, Loosemore, and Hanby (1971) which was inspired by that in Rokkan (1968) but shall simply show their results (see Table 1). We have added to the four systems they analyzed, threshold values for three other systems[14]): the cumulative vote, the limited vote with K votes cast, $K < m$ (used in Japan with $K = 1$), and the modified St. Lague, the one which uses an initial divisor of 1. 4.

Clearly, if there are more parties, we see from Table 1 that the threshold of representation is lower than the threshold of exclusion for all seven election systems. We also see that, while there is some duplication of T_E values, each of the seven systems has a unique value for T_R[15]). The threshold of representation is lowest for the St. Lague system, highest for plurality. The threshold of exclusion is highest for plurality and lowest under the St. Lague and Greatest Remainder systems. *The implications of this for representation of minorities are not, however, quite what they might at first seem to be.* We must extend our analysis to the legislature as a whole!

For simplicity, assume an r-member legislative body divided into $\frac{r}{m}$ districts of m members each. The national threshold of representation for each of our voting schemes is simply $T_R\left(\frac{m}{r}\right)$. If we compare T_R for plurality $\left(= \frac{1}{nr}\right)$ with T_R for D'Hondt $\left(= \frac{m}{(m+n-1)r}\right)$ we may readily show by simple algebra (Rae, 1971, p. 158) that the threshold for plurality is *always* lower than that for D'Hondt. This implies that, when minorities are geographically concentrated, a plurality system may be more favorable to at least initial minority representation than the most common "PR" system. A similar surprising result holds when we compare national thresholds of representation for plurality and the limited vote. On the other hand, plurality's national threshold of representation is the same as that of St. Lague and is sometimes higher and sometimes lower than that for the remaining schemes listed in Table 1.

[14]) We have also extended their analysis in a straightforward fashion by including the bloc vote, whose threshhold properties are identical to those of smd plurality.
[15]) Of course, for certain values of n, m, and k, T_R values will be identical for two or more of our schemes.

Table 1

Threshold Values in m Member Districts as a Proportion of Total District Vote

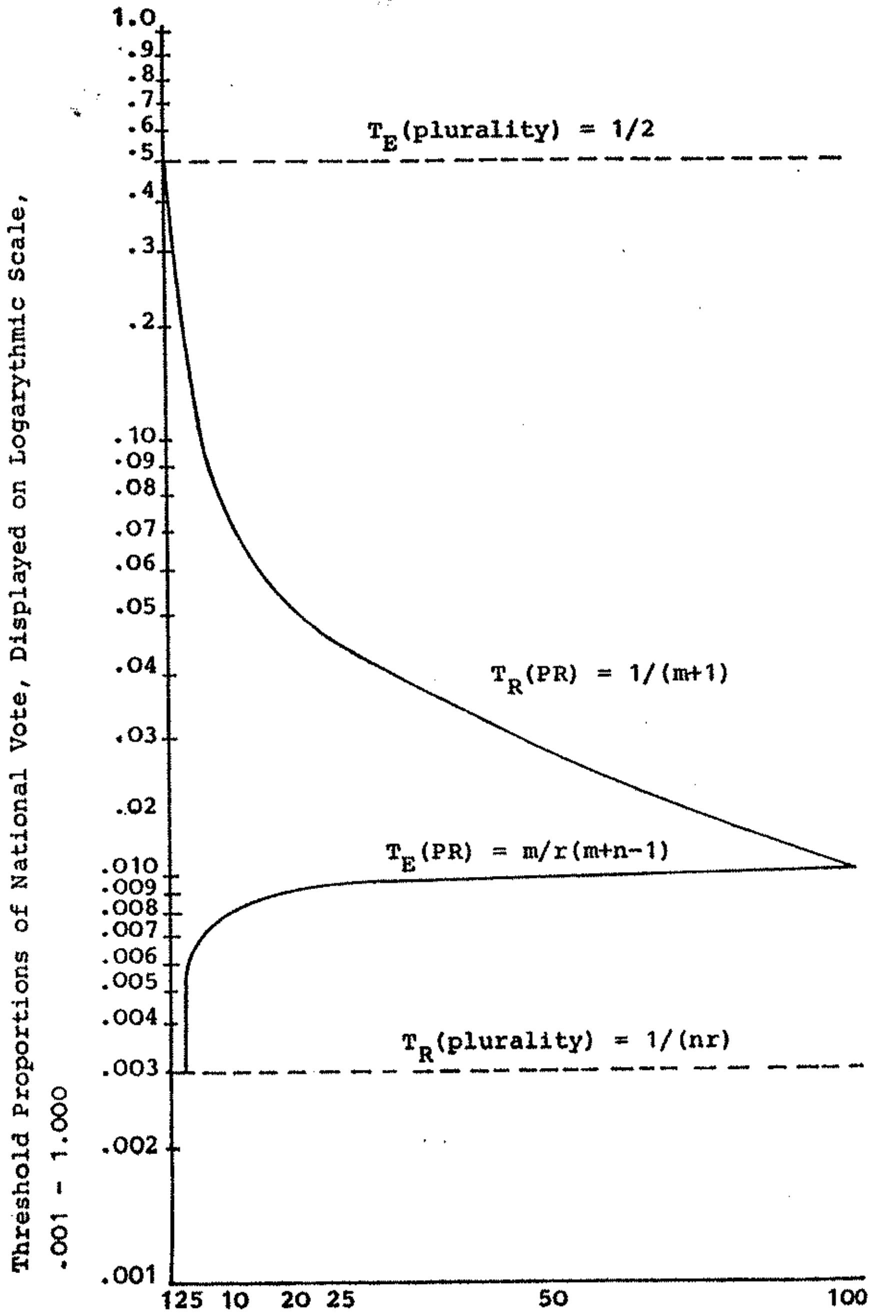
	Plurality and Bloc Vote	d'Hondt	St. Lague
T_E	$1/2$	$\frac{1}{m+1}$	$\frac{1}{2m}$
T_R	$1/n$	$\frac{1}{m+n-1}$	$\frac{1}{2m+n-2}$
$T_E - T_R$	$\frac{n-2}{2n}$	$\frac{n-2}{m^2+mn+n-1}$	$\frac{n-2}{2mn}$

	Largest Remainder	Limited[a] Vote ($k < m$)	Cumulative Vote[b]
T_E	$\frac{1}{2m}$	$\frac{k}{k+m}$	$\frac{1}{m+1}$
T_R	$\frac{1}{mn}$	$\min\left(\frac{1}{n}, \frac{k}{k+m+n-2}\right)$	$\frac{1}{mn-m+1}$
$T_E - T_R$	$\frac{n-2}{4m^2+2mn-2}$	$\frac{k(n-2)}{(k+m)^2+(k+m)(n-2)}$ or $\frac{kn-k-m}{(k+m)n}$	$\frac{mn-2m}{m^2n-m^2+mn+1}$

	Modified St. Lague
T_E	$\frac{1.4}{2m+.4}$
T_R	$\frac{1.4}{2m+1.4n-2.4}$
$T_E - T_R$	$\frac{1.96n-3.92}{(2m+.4)(2m+1.4n-2.4)}$

Partial Source: Table 1 (Rae, Loosemore, and Hanby, 1971, p. 485).
[a] The largest party is assumed to divide its vote equally among all its candidates to provide a more realistic assessment for T_R. Otherwise, T_R could be as low as one vote ($k > 1$) when all of a party's vote goes to one of its candidates.
[b] We assume each party runs a full slate of m candidates.

Figure 4



Threshold Curves for D'Hondt Compared with Plurality Threshold Points ($r = 100$, $n = 3$, $m = 1$—100, Vertical Scale Logarythmic).

If, now, we turn to exclusion thresholds for the nation as a whole, we find that T_E for plurality remains $\frac{1}{2}$ and, if we assume an even distribution of party strength across districts, for an insurgent party confronted with a strong established party (although not necessarily the same one) in each district, it is clear (Cf. Rae, 1971, pp. 162—163) that the other national thresholds of exclusion also remain as given in Table 1. Figure 4 (identical to Rae, 1971, Figure 10. 2, p. 160) shows the comparison of T_R and T_E for plurality and the D'Hondt rule.

> "The effect of (the D'Hondt Rule) is to 'raise the floor and lower the ceiling' as T_R increases with m and T_E declines until — in the case $m = r$ — they meet, and the importance of spatial distribution disappears. As districts grow larger, the relationship between a party's vote and its chance for representation becomes increasingly determinate." (Rae, 1971, p. 164.)

Analogous assertions (at least with respect to T_E) can be made in comparing plurality with each of our other voting schemes. The five schemes we have dealt with make it harder to completely exclude minority parties from the national legislature than does the first-past-the-post system.

A common approach to the seats-votes relationship has been the examination of the "distortion" caused by particular voting schemes, i. e., the extent to which electoral groupings are under/overrepresented in terms of seats in proportion to their vote percentages. A direct indication of this distortion is given by the index D:

$$D = \frac{1}{2} \sum_i | v_i - s_i |$$

where s_i is the seats for party i, the bars indicate we are to look at absolute values, and the multiplier $\frac{1}{2}$ is inserted to produce an index which ranges from 0 to 1. (Loosemore and Hanby, 1971, pp. 468—469.)

Rae (1967, pp. 84—85) has looked at a closely related index, average deviation, defined as $\frac{2D}{n}$, where he has found, for 116 postwar European elections, an average deviation of only 2.39 %. Rae (1967, p. 87) in comparing this figure with average inter-election shifts in popular vote, asserts that the effect of election systems upon the competitive positions of political parties is "marginal in comparison to the effect of election outcomes". As we would expect, Rae finds that average deviation is higher for elections under plurality as compared with those under party list systems (see Rae, 1967, Chap. 5).

Loosemore and Hanby (1971) have calculated the maximum values of D for four election systems as a function of m and n. We have extended their analysis to include cumulative vote and the limited vote. Results are shown in Table 2.

Table 2

The Index of Distortion D, as a Function of District Size and Number of Parties Contesting the Election

Plurality and Bloc Vote	d'Hondt	St. Lague	Largest Remainder	Limited[a] Vote ($k < m$)	Cumulative Vote[b]
$1 - 1/n$	$\frac{n-1}{m+n-1}$	$\frac{n-1}{2m+n-2}$	$\frac{1}{m}(1-\frac{1}{n})$	$\frac{kn-k}{kn-k+m}$	$\frac{n-1}{m+n-1}$
Modified St. Lague					
$\frac{1.4\,(n-1)}{2m+1.4n-2.4}$					

Partial Source: Loosemore and Hanby (1971, Table 4, p. 475).

[a] Each party is assumed to divide its votes equally among all its candidates.

[b] Minor parties are here assumed to concentrate their votes on one candidate. If votes were divided equally among all candidates, the cumulative vote would have the same index of distortion as Plurality.

Loosemore and Hanby (1971, p. 475) found that the system with the least distortion was the largest remainder scheme ("Under largest remainder, any constituency with a large number of seats (say 10) is guaranteed nearly perfect proportionality"); the next lowest is D'Hondt, and the greatest distortion index (rapidly approaching one as n increases) occurs under plurality. We would add to those findings that the cumulative vote is identical in its index of distortion to the D'Hondt scheme, that the modified St. Lague is intermediate in distortion between D'Hondt and St. Lague, and that the limited vote should be placed intermediate in distortion between plurality and D'Hondt (for $K = 1$, the Index of Distortion for the limited vote is identical to that of D'Hondt, for $K > 1$ it is invariably higher, as some simple algebra will quickly reveal)[16].

Loosemore and Hanby (1971) provide tables which show values of D for various values of m and n for the four schemes they consider but we shall not reproduce them here since they add little that can't be seen directly from inspection of the entries in Table 2[17].

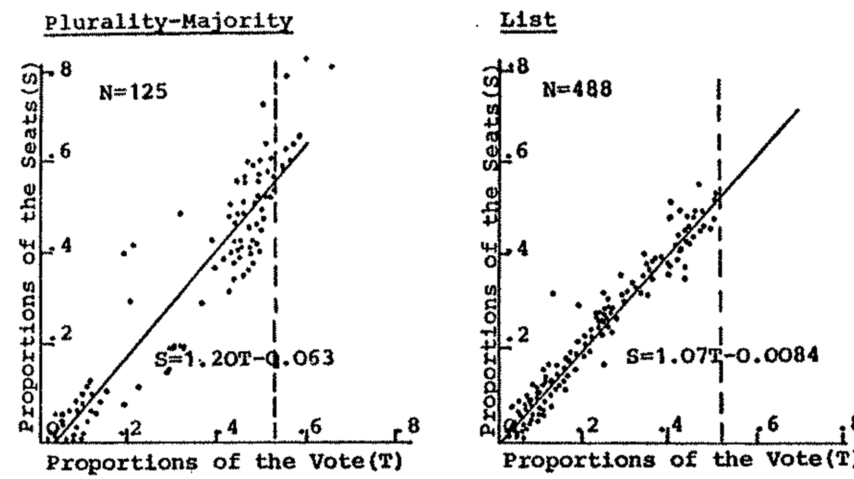
One further empirical point. Basing their conclusion on data drawn from Rae (1967), Loosemore and Hanby (1971, p. 477) assert that party list systems exhibit an actual distortion which is only about 20 % of their potential maximum distortion.

[16]) Cf. Blondel (1970, p. 191).

[17]) For example we see that the distortion index for St. Lague is more affected by m than is the distortion index for d'Hondt.

A fourth approach to understanding the seats-votes relationship is graphical, the seats-votes curve. This has customarily been used as a descriptive device to depict deviations from strict proportionality. We have reproduced the scatterplots from Rae (1967, Table 5. 1, p. 96) which give, for a number of postwar European elections, the observed national seats-votes intersections for each of the parties in those elections, controlling for type of election scheme (majoritarian vs. party list). See Figure 5.

Figure 5



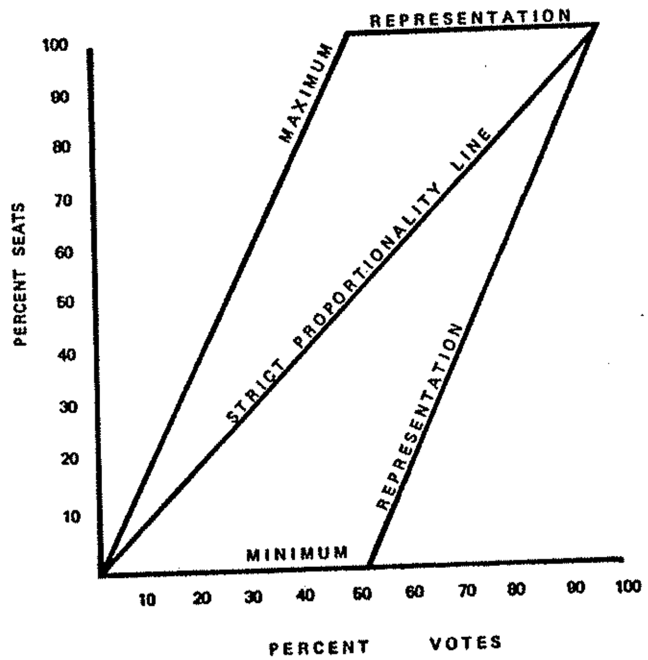
Proportions of the Seats (S) Plotted Against Proportions of the Vote (T): Plurality-Majority Formulae Compared with List Systems.
(Source: Rae, 1971, p. 90.)

As we see from these graphs, the points cluster *much* more closely about the proportionality line for the party list elections than for the plurality-majority elections. Rae has used ordinary least squares to fit regression lines, but we regard his regression coefficients (1.20 for plurality vs. 1.07 for party list) as being quite misleading, at least in the plurality/majority case. Inspection of Figure 5 reveals that majoritarian elections' scatter points are not well fitted by a straight line; an S-shaped curve would give a much better fit. (More on this point below.)

Another way to make use of the seats-votes curve is to graph maximum/minimum theoretically possible seat percentages as a function of percent votes achieved and type of election system. As far as we are aware, this paper is the first in which this device has been used. Figure 6 shows for a national election via plurality, $n = 2$, the maximum/minimum percentages[18]). The area bounded by these two curves can be thought of as the *Coefficient of Distortion*, analogous to the Gini Index for a Lorenz curve (see Alker, 1967). The Coefficient of Distortion so obtained is directly related to Hanby and Loosemore's (1971) Index of Distortion but we shall not attempt to specify the exact nature of the mathematical link here.

Figure 6

Graph of Theoretical Maximum/Minimum Vote Percentages in a Plurality Election with $n = 2$



It's easy to see that for two party plurality elections, the Coefficient of Distortion is $\frac{1}{2}$. Analogous figures could be constructed for our other

[18]) We have specified the curves for $r = \infty$. Finite values of r would introduce discontinuities but would not affect the basic relationships expressed. To calculate points on the minimum representation line we solve the equation

$$vr = 50\,(r - s) + 100s$$

obtaining

$$s = \frac{(v - 50)\,r}{50}$$

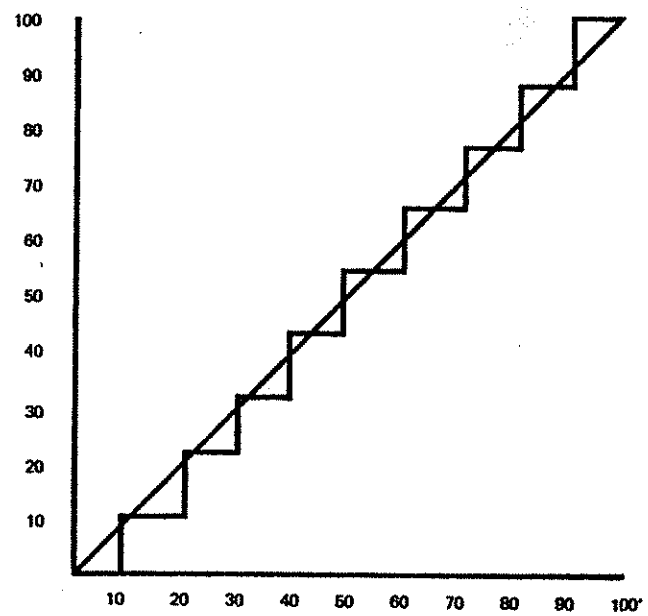
Expressing s as a percent of r, we have

$$\frac{s}{r} = \frac{v - 50}{50}$$

voting schemes. In Figure 7 we show the graph of theoretical maximum/minimum vote percentages for a D'Hondt district election with $n = 2$, $m = 9$[19]). We may show that the Coefficient of Distortion in this case is only .07[20]).

Figure 7

Graph of Theoretical Maximum/Minimum Vote Percentages in a District Election with $n = 2$, $m = 9$ D'Hondt Rule



Another analytic tool for understanding the seats-votes relationship for various voting schemes is to postulate some probability distribution of party strength across districts and then calculate the *expected* distortion

[19]) Note that for $n = 2$ for the D'Hondt scheme T_R and T_E are identical.

[20]) For those unwilling to count squares, we note that the formula for the Coefficient of Distortion in the D'Hondt case can be shown to be (Grofman, 1975c):

$$CD = \sum_{i=1}^{m} \left(\frac{i}{m+n-1} - \left(\frac{i-1}{m} \right) \right)^2 + \sum_{i=1}^{m} \left(\frac{i}{m} - \frac{i}{m+n-1} \right)^2$$

which for $n = 2$, $m = 9$, reduces to

$$CD = 2 \sum_{i=1}^{m} \left(\frac{i}{m\,(m+1)} \right)^2$$

This formula can be shown to be convergent. (Cf. Morrey, 1962, Theorem 17.8 p. 453). Hence, as we would expect for the D'Hondt Rule, the Coefficient of Distortion approaches 0 as $m \to \infty$.

or misrepresentation which such a distribution would imply. Lewis Carroll (the mathematical author of *Alice in Wonderland)* did just that almost a century ago (1884) for the limited vote and bloc vote, but his work remained unknown or misunderstood until only a few years ago when the economist Duncan Black, who is also an authority on Carroll, restated Carroll's arguments and calculations in a clearer form (Black, 1967)[21]).

Carroll assumed $n = 2$ and that each party had sufficient information about its probable vote support to run the "optimum" number of candidates and that each party was able to evenly divide its votes among its candidates[22]). For example, in a 3- seat 2-vote district, if one party has less than 50 percent it should run two candidates; if it has more than 60 percent it should run 3. With over 60 percent of the vote, a party must, under our assumptions, win all 3 seats. With 50 + to 60 — percent of the vote a party must win 2 seats if it contests 2 seats. If it contests 3 seats, it will win only 1 seat if the other party contests two seats. With 40 + to 50 — percent of the vote, a party should contest 2 seats but expect to win at most 1 of them. In such a case, if the other party errs and runs more than its optimum number of candidates, running the extra candidate will pick up an extra seat, and it can't *ever* hurt. With less than 40 percent of the vote it can't hurt to run 2 candidates, even though, unless you've misestimated, the situation is hopeless[23]).

Carroll then goes on to assume that the vote distribution is rectangular, i. e., all divisions of the vote are equally likely. Hence, for example, in the 3- seat 2-vote case there is a 40 percent probability that the vote percentage for party 1 will be in the 0—40 percent range, a 10 percent probability that it will be in the 40—50 percent range, a 10 percent probability that it will be in the 50—60 percent range, and a 40 percent probability that it will be in the 60—100 percent range. Thus, there is an expected distortion (D) of roughly

$$[(|.20 - 0| + |.80 - 1.00|).4 + (|.45 - .33| + |.55 - .66|).1$$
$$+ (|.55 - .66| + |.45 - .33|).1$$
$$+ (|.80 - 1.00| + |.20 - 0|).4]/2 = .163$$

[21]) Cf. Black (1957, p. 191) which, as Black (1967, p. 17) notes, involves a misstatement of Carroll's argument.

[22]) In a two party race under the limited vote and under the information assumptions specified, an "optimal" strategy is easy to calculate; run exactly ℓ candidates ($k < \ell \leq m$) for the maximum ℓ for which your expected vote percentage, v, is such that $\frac{vk}{\ell} > \frac{(1-v)\,k}{m-\ell}$ for $m - \ell \geq k$; or such that $\frac{vk}{\ell} > 1 - v$ for $m - \ell < k$.
If there is no ℓ for which these inequalities are satisfied your situation is hopeless! Run k candidates, you've nothing to lose.

[23]) Recall that Carroll assumes that the vote percentages are accurately predictable.

Carroll, however, uses an Index of Nonrepresentation which is subtly different from D. Carroll lacks the sum over all parties of the absolute value of the deviation between the vote necessary to guarantee the winning of S seats (for some particular voting scheme) and the actual vote which results in the S seats being won. Thus, for example, in a 3- seat 2-vote limited vote, two-party district election, if v_i was .45, Carroll's Index of Nonrepresentation would be .10 ($= |.45 - .40| + |.55 - .50|$), since 40 percent of the vote would have been sufficient to guarantee the one seat won by the party with 45 percent and 50 percent would have been the percentage needed to guarantee the two seats won by the party with 55 percent of the vote[24]). Under the assumption of a rectangular distribution, the expected value of Carroll's Index of Nonrepresentation in the 3- seat 2-vote 2 party case is

$$(|.20 - 0| + |.80 - .60|).4 + (|.45 - .40| + |.55 - .50|).1$$
$$+ (|.55 - .50| + |.45 - .40|).1 + (|.80 - .60| + |.20 - 0|).4$$
$$= .34$$

Hence, the expected proportion of voters represented is .66[25]). Carroll then calculates the mathematical expectation of the percentage of the electorate represented for $1 \leqslant m \leqslant 6$, $K \leqslant 1$ (see Table 3). (Note that for the limited vote if $K = m = 1$, we have simple plurality. If $K = m > 1$, we have the bloc vote. Thus, plurality and the bloc vote can be thought of as special forms of the limited vote.) We see from Table 3 that the greatest representation occurs for $K = 1$, although differences between neighboring values of K decline as m increases. Carroll advocated the single nontransferable vote, but also was interested, quite naturally, in the question of optimum district size.

Table 3
Expected Percentage of the Electorate Represented for the Limited Vote, the Bloc Vote, and Plurality Under the Assumption of a Rectangular Distribution: $n = 2$, $1 \leq m \leq 6$, $k \leq m$

m/k	1	2	3	4	5	6
1	.51					
2	.68	.51				
3	.76	.66	.51			
4	.81	.74	.64	.51		
5	.84	.79	.71	.61	.51	
6	.86	.81	.74	.66	.59	.51

Source: Black (1967, Figure 1, p. 2).

[24]) D for this case, would be $1/2\,(|.45 - .33| + |.55 - .66|) = .116$.

[25]) For an alternative and much more elegant formula for deriving the expected value for Carroll's Index see Black (1967, pp. 8—15).
For voting schemes which are reasonably proportional in their translation of votes into seats, Carroll's Index will be roughly equal to twice D.

"The change from single member to two-member districts changes the percentage of unrepresented electors from 49 to 32 ...; whereas the change from 5-member to 6-member districts only changes the percentage from 16 to 14 ... The conclusion is that *the* important point is to have as few single member and even as few 2-member districts as possible; but that, when we have got as far as to districts returning 4 or 5 members each it is hardly worthwhile to go further." (Carroll, 1884, pp. 25—6, cited in Black, 1967, p. 16; emphasis in original.)

Clearly, we could, in principle, readily generalize the results of Table 3 by looking at distributions other than the rectangular (e. g., the normal) and/or by looking at the Index of Distortion rather than Carroll's Index of Nonrepresentation. We shall not, however, pursue these issues further[26]).

We now turn to one of the most famous conjectures about a social science lawful relationship, the "cube law". First conjectured over 70 years ago by a mathematician named McMahon (perhaps inspired by Edgeworth, 1898), the "Law" was popularized by J. P. Smith in testimony before the Royal Commission of Electoral Systems (1910, p. 81). Smith asserted that in a two-party plurality election with single member districts, if votes were divided between the parties in the ratio $\frac{V}{1-V}$, seats would be divided in the ratio $\frac{V}{1-V}^3$, i. e., if S is the seat proportion won by party 1 and V is its national vote proportion, the cube law is

$$\frac{S}{1-S} = \frac{V}{1-V}^3 \tag{1}$$

Some simple algebra reveals that expression (1) can be recast as

$$S = \frac{V^3}{3V^2 - 3V + 1} \tag{2}$$

The cube law is much like Fermat's famous last "theorem". Just as a great deal of (so far fruitless) energy has gone into seeking proofs/disproofs of this theorem, so (especially recently) has there been a great deal of interest in the purported universality of the cube law.

26) Recall, however, that we earlier showed that, for the limited vote, the Index of Distortion took its minimum value when $k = 1$. When $k = 1$ and $n = 2$, the Index of Distortion for the limited vote is $\frac{1}{m}$, which decreases in m at a decreasing rate. Recall also that T_E for the limited vote is $\frac{k}{k+m}$, in this case = .40.

law was rediscovered, and an attempt made to relate it to statistical considerations about the distribution of vote strengths across constituencies in a now classic article by Kendall and Stuart (1950). Kendall and Stuart (1950) point out the distribution function defined by the cube law is almost identical to the distribution function of a normal distribution with mean .5 and variance .0187. Further statistical considerations were introduced by March (1957—58), and Coleman (1964). Both sought to account for the fact that the observed variance in vote proportions in British and U. S. national legislative contests was much lower than what would be expected were party strength normally distributed across election districts with a mean value of p and variance $= p\frac{(1-p)}{N}$. Both March (1957) and Coleman (1964) hypothesized group pressure models which might create lopsided majorities for the dominant party in some districts, but only Coleman offers an actual model which he uses (with considerable success) to fit data from U. S. trade union elections.

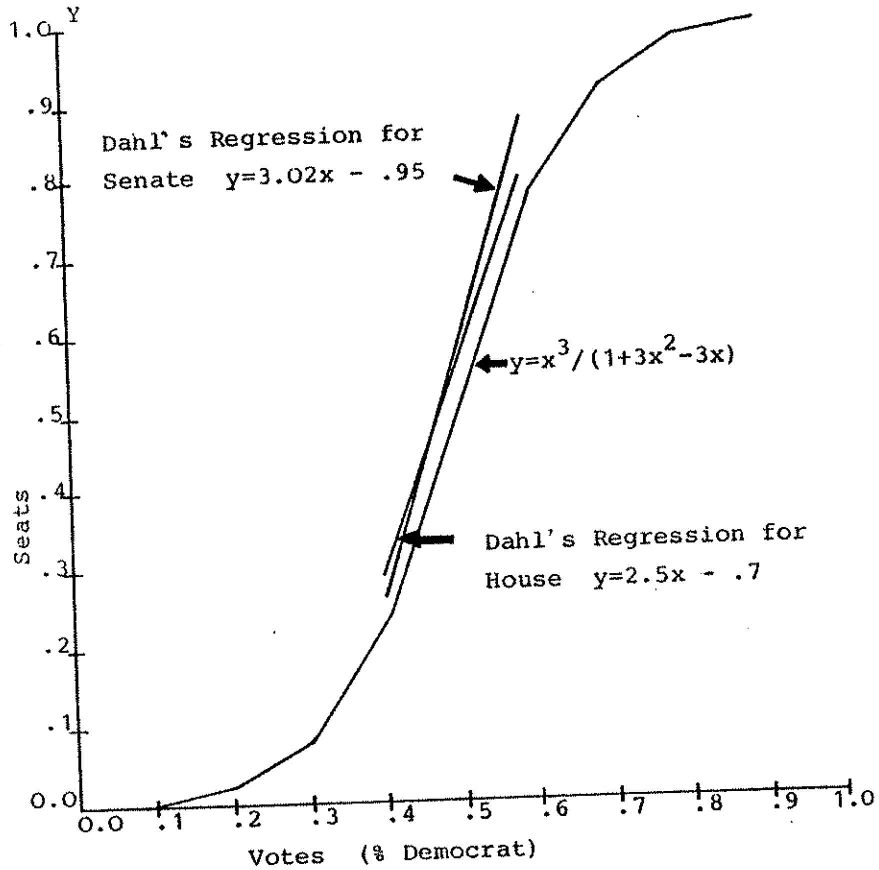
Empirical work on the fit of the cube law to British, U. S., and Canadian election data has been done by a number of authors. (See, e. g., Kendall and Stuart, 1952; Dahl, 1956; Butler, 1953; Qualter, 1968; Spafford, 1973). March (1957—58) has shown that the cube law in the range .40 — .60 can be approximated by the straight line $S = 2.808V - .904$. Least square techniques of varying degrees of statistical sophistication have been used to find the best-fitting straight line for series of plurality elections. For five British elections, March (1957—58) found a regression line with equation $S = 2.77V - .87$. Using U. S. Congressional elections (1928—1954) and U. S. Senate elections (1928—1952), Dahl (1956, pp. 148—149) found equations of $S = 2.5V - .70$ and $S = 3.02V - .95$ respectively. We show Dahl's results and the cube law relationship in Figure 8 (reproduced from Benson, 1972, p. 77). These regressions seem to suggest a good fit for cube law predictions in the observed range.

Following Theil (1969, 1970), Tufte (1973) uses a logit model of the form

$$\log_e \frac{S}{1-S} = B_0 + B_1 \log_e \frac{V}{1-V} \tag{3}$$

If $B_0 = 0$ and $B_1 = 3$, this is identical to the cube law. Thus, fitting B_0 and B_1 provides a direct test of the cube law. Moreover, the coefficients B_0 and B_1 have clear empirical import. The first is a measure of bias, the extent to which one party must gain more than 50 percent of the votes to get 50 percent of the seats; the second is a measure of the "swing ratio", the proportion of seats "earned" by each percent increment in national vote. Tufte's is clearly the most comprehensive empirical study

Figure 8



The Cube Law of Representation and Dahl's Regressions for the U.S. Congress. (Source: Benson, 1973, Figure 3—16, p. 77 [from Dahl, 1956]).

of the cube law to date and his results are negative. He finds a wide variation in the swing ratio, with only Great Britain approximating the magic value of 3, and he also finds considerable bias. (We have reproduced his results as Table 4.) Tufte also finds that in U. S. Congressional elections the swing ratio appears to be *drastically* declining, with a value of only .71 in the period 1966—1970 with a very high bias (7.9 %) in favor of the Democrats (Tufte, 1973, Table 6, p. 550). Thus, a vote increment equal to seven percent of the national vote would be expected to yield

Table 4

Testing the Predictions of the Cube Law (and Simultaneously Estimating the Logit Model)

	β_0	β_1	Standard error of slope	r^2	Does $\beta_0 = 0$ and $\beta_1 = 3$ as cube law predicts?	Is $\beta_0 \neq 0$; that is, is there a significant bias?
Great Britain	—.02	2.88	.30	.94	yes	no bias
New Zealand	—.12	2.31	.27	.91	no	yes, there is a bias
United States, 1868—1970	.09	2.52	.24	.68	no	yes
United States, 1900—1970	.17	2.20	.15	.86	no	yes
Michigan	—.17	2.19	.43	.76	no	yes
New Jersey	—.77	2.09	.59	.29	no	yes
New York	—.23	1.33	.19	.74	no	yes

Source: Tufte (1973), Table 2, p. 546.

an increase in seats gained equal only to 5 percent of the seats being contested[27]). In some instructive histograms, Tufte (1973, Figure 7, p. 553) has demonstrated how in the U. S. Congress the distribution of party strength across districts has dramatically changed from a nearly normal distribution centered around .50 (but plus a huge Southern Democratic "tail" of virtually uncontested districts) to a bimodal distribution in which competitive districts (those in the vote range .40 — .60) are becoming scarcer and scarcer (and with a diminishing but still quite large Southern Democratic tail). Tufte has shown that, for certain plausible party strength distribution assumptions, the "more uniform electoral swings are across the nation, the greater will be the swing ratio" (Tufte, 1973, p. 547). In other words, the more shifts in popular vote hinge on national rather than local issues, the greater will be the swing ratio. Tufte

[27]) Tufte (1973, pp. 551—553) attributes much of this change to recent redistricting which, according to him, has allowed parties to fashion districts. This view has been disputed by Walter Dean Burnham in an exchange of letters with Tufte (Burnham, 1974). Tufte attributes the high bias in favor of the Democrats to the many low turnout Southern election districts which regularly go Democratic. Thus, for the same votes cast, the Democratics gained more seats than did the Republicans who won in districts with higher turnout (Tufte, 1973, p. 548). We believe this point is not properly dealt with by Tufte, since turnout effects are not independent of distribution effects, eg. low turnout districts tend to be districts with highly skewed distributions of party support. This point we hope to clarify in work in progress (Grofman, 1975a), for our disagreements with Tufte may really be more semantic than substantive.

then hypothesizes that we would expect Britain, with a tighter national party organization than the U. S., to have a higher swing ratio than the U. S., which we indeed find to be the case (see Table 4). While I find this hypothesis plausible, it seems to me that of more direct relevance to the swing ratio is the variance of the distribution of party strength across electoral districts. Tufte is clearly quite sensitive to this point (Cf. Tufte, 1973, pp. 547—554, esp. 547) but I still do not think he gives it the importance it deserves.

For example, Tufte (1973, p. 547) asserts that "in U. S. Congressional elections the swing ratio will be greater in on-year elections with the presidential contest on the ballot than in off-year elections when national forces are somewhat diminished", and finds that "those expectations are borne out in both cases". Tufte attributes this to a greater national uniformity in shifts in party allegiance in presidential years, but we believe it can better be understood in terms of the relationship between turnout and the skewedness of district party vote strengths, where low turnout (off-year elections) is associated with high skewedness. (See our comments in note 27.) Although I differ somewhat with Tufte as to cause of the changes in swing ratio and bias, I regard his treatment of the data as both thorough and ingenious. For example, he has cleverly used U. S. Congressional election data, in the period 1952—1970, and some simple algebra to show that

> "In order to regain its seats lost in the previous on-year election, the out-party needs a shift almost one-fourth (.24) greater than the shift in votes which originally won those seats for the President's party. For example, if in an on-year election the President's party gained four percentage points in votes over its previous winnings ... then the out-party would need a vote shift of 1.24 × 4 % = 5 % to regain the lost seats." (Tufte, 1973, p. 548.)

Clearly, the swing ratio and system bias of any plurality system varies with the distribution of party strengths across districts. Let us look at extreme cases: If we had two purely sectional parties (e. g., 50 districts of 80 % — 20 % and 50 districts of 20 % — 80 %) the bias B_1 would be zero and the swing ratio, B_2, quite low (at maximum .3 and probably more like .03). On the other hand, if one party were more sectional than the other (e. g., 25 districts of 80 % — 20 % and 75 districts of 40 % — 60 %) then the more sectional party would be strongly biased against (B_1 = —.25) and the cube law would again be inapplicable. If party strength were normally distributed across districts, then the swing ratio would be very large since a party that won even slightly more than half the votes would win all the seats (see Figure 6). Taagepera (1973) has generalized the notions of swing ratio and bias to other than plurality systems and, inspired by Theil (1969, 1970), has proposed an interesting generalization of the cube law.

Theil (1969) has shown that, if the functional relationship between seats-votes and votes is to satisfy certain desirable mathematical properties (the nature of which we shall not attempt to go into here, except to note that one of them is generalizability to the n-party case), then it must be of the form

$$\frac{S_A}{S_B} = \left(\frac{V_A}{V_B}\right)^q \tag{3}$$

where S_A (V_A) is the seat percentage (vote percentage) for party A, and q is simply some unknown exponent.

Taagepera (1973) has proposed the following generalized seats-votes law for non-biased elections[28])

$$\log S \cdot \log \left(\frac{S_A}{S_B}\right) = \log V \cdot \log \left(\frac{V_A}{V_B}\right) \tag{4}$$

where V is total vote and S is the number of subdivisions in which the party plurality wins all the seats. In the case of Anglo-Saxon Parliamentary elections, S = number of seats in the legislature. In the case of the U. S. Electoral College, currently S = 51; each state (+ D. C.) is such a subdivision. In the case of direct presidential elections, S = 1; the whole country is a subdivision. In the case of list systems, STV, and other "proportional" schemes, Taagepera sets S = V. As Taagepera (1973) puts it "ideal proportional elections are effectively equivalent to such one-voter constituencies, since in such elections there is no larger subdivision in where the party with plurality would (invariably) carry all the seats".

Some simple algebraic manipulation of expression (4) reveals that if this relationship is to hold we must have

$$\frac{\log S}{\log V} = \frac{\log \left(\frac{V_A}{V_B}\right)}{\log \left(\frac{S_A}{S_B}\right)} \tag{5}$$

In other words the plot of $\frac{\log S}{\log V}$ vs. $\frac{\log \left(\frac{V_A}{V_B}\right)}{\log \left(\frac{S_A}{S_B}\right)}$ should be a straight line.

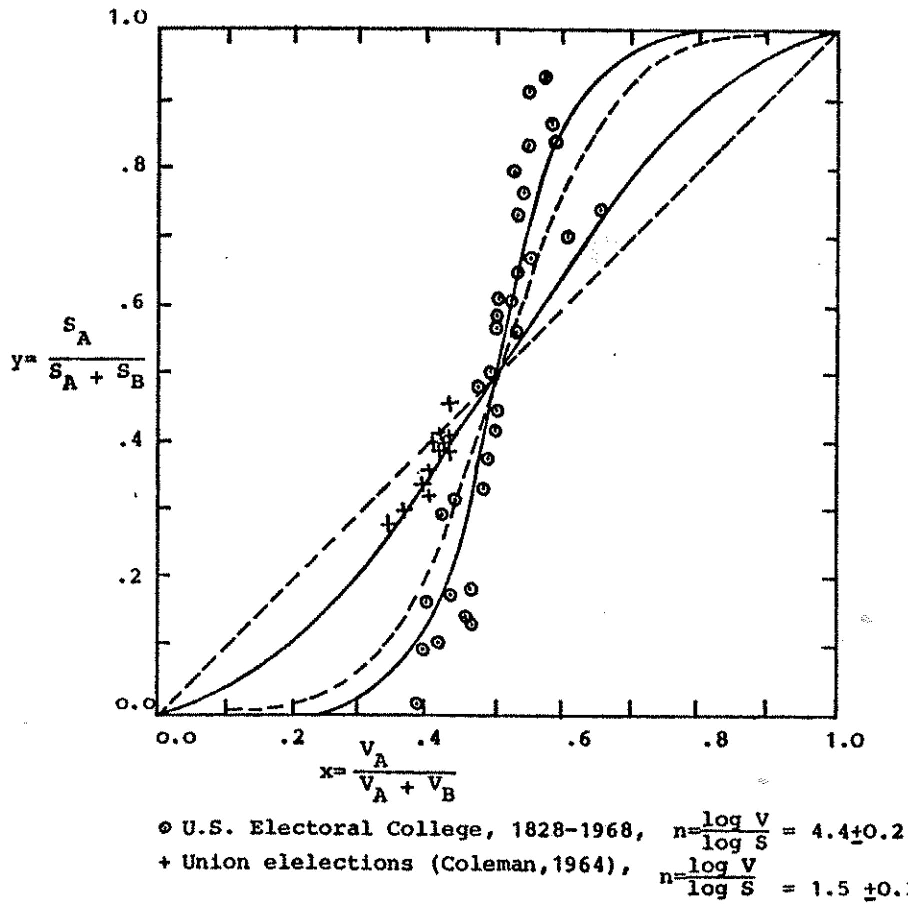
[28]) Taagepera (1973) has dealt with the bias factor but we omit his treatment of it. The equation we present below is analógous to that for plurality elections with B_1 set equal to 0.

Furthermore, for the cube law to hold, for simple plurality elections $\frac{\log S}{\log V}$ must equal $\frac{1}{3}$. Taagepera (1972) has proposed a "cube root law of assembly sizes" based on a theoretical model of multi-level communication channels, which results in the hypothesis that the size of the legislature will be proportional to the cube root of the politically active population. For purposes of operationalization, Taagepera (1972) took the politically active population as equal to the literate adult population and found his cube root law to be a reasonable though far from perfect approximation to contemporary data. Similar results were found by Dahl and Tufte (1973, p. 81—84) who used total population rather than total literate adult population[20]). Taagepera (1973) shows that if the number of voters is roughly 2/3 the literate adult population, then his cube root law of assembly size will result in the cube law for the seats/votes relationship. Taagepera (1973) has plotted $\frac{\log S}{\log V}$ vs. $\frac{\log\left(\frac{V_A}{V_B}\right)}{\log\left(\frac{S_A}{S_B}\right)}$ for the U. S. Electoral College 1828—1970 (using an averaging procedure to collapse the data into one point); for national legislative elections in the U. K., New Zealand, the U. S. and Canada; and for elections in the U. S. Typographer's Union which had previously been studied by Coleman (1964). His results are shown in Figure 9, and Figure 10 shows the direct plot of seats vs. percent votes based on his results. As we see, except for one of the union elections, the points cluster reasonably well around his predicted seat-vote equation line. We find Taagepera's own summing up of his findings to be worth citing.

"The data shown in Figures 9 and 10 by no means establish the validity of the seat-vote equation. Many data sets could be found that would deviate widely from the prescribed pattern. Even the data sets shown could be better expressed by some empirical formula involving coefficient values determined empirically for the particular data involved. (Of course, the more empirical coefficients one adds, the better the fit, but at the expense of a loss of generality.) But

[20]) Dahl and Tufte (1973, p. 81) found that "countries with rapidly increasing populations tend to have smaller parliaments than predicted by their absolute size". This could be accounted for in terms of Taagepera's model since such countries are also likely to have a lower than average percent literate population. Dahl and Tufte (1973, p. 81) also found that "countries with multiparty systems tend to have larger parliaments (other things being equal, two-party systems have parliaments averaging about 137 seats; multiparty systems, 195 seats)".

Figure 9

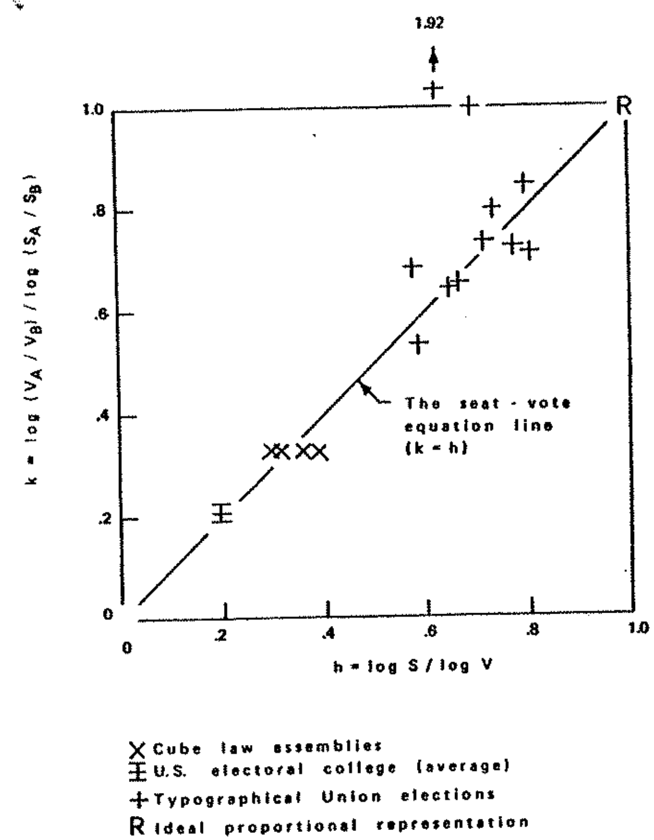


The Relative Number of Seats and Votes. Data sources as in Figure 10.
(Source: Taagepera, 1973).

Figures 7 and 8 demonstrate the plausibility of the seats/votes equation and its ability to approximate a wide range of election results *without resort to any arbitrary coefficients.*" (Taagepera, 1973, with some change in notation, emphasis in original.)

Two other recent papers on the cube law are also worth citing. Casstevens and Morris (1973), using data from the U. S. House and Senate, have found a modified form of the cube law to be useful at predicting the

Figure 10



percent of motions which will be passed in a legislature as a function of the average vote division in that body. Sankoff and Mellos (1972) have used game theory to investigate a situation in which each party chooses an optimal investment of campaign resources in ignorance of its opponent's pattern of expenditure but in awareness of the relative magnitudes of resources available to each party, and of which investments determine (in marginal cases) the election outcomes. The swing vote is shown to depend, i. a., upon the percentage of "hard core" voters of each persuasion in each constituency, i. e., those not temptable to cross over to the other side. Under certain not directly testable assumptions about the size of the "hardcore", a cube-law is obtained, although Sankoff and Mellos (1973) appear to opt for a "square law" as the most plausible prediction.

One other point on the seats-votes relationship: Taagepera (1973) has pointed out that that relationship may be directly manipulated by using the seat assignment rule, $\frac{S_A}{S_B} = \left(\frac{V_A}{V_B}\right)^q$. By choosing q appropriately, any desired under/over representation of minorities/majorities can be obtained[30]).

Rae, Loosemore, and Hanby (1971) draw two interesting conclusions from their analysis which can be extended to hold true for the two additional cases we have introduced:

1. If two or more parties wish to minimize the representation of a third party, there is incentive, under all seven systems, for collusion, i. e., "district trading". (The logical argument underlying this assertion is developed at some length in Rae et al. (1971); arguments analogous to theirs can be constructed for modified St. Lague, the cumulative and the limited vote.)
2. For each of the multi-member voting schemes, thresholds of exclusion are inverse functions of district size (m); furthermore, the thresholds decrease at a decreasing rate as m increases. Thus, we may manipulate the representation of minority parties by choice of voting scheme and/or by manipulation of m. Inspection of Table 1 readily reveals that although, e. g., for fixed m, the greatest remainder exhibits a lower exclusionary threshold than does the D'Hondt system, this result is reversed if we compare a D'Hondt system in one district to a St. Lague system in another district with half as many seats[31]).

[30]) Of course the seats-votes relationship may also, as Taagepera (1973) is careful to point out, be manipulated by reducing/increasing S, a manipulation equivalent to reducing/increasing m in our earlier discussion of the Loosemore & Hanby (1971) results.

[31]) If we are interested in the likely consequences of changes in election systems we can, of course, attempt to simulate them either by using actual election data or by using theoretical data generated by some stochastic process. The former approach is used by Scheuch and Wildenmann (1965) and by Butler et. al. (1958—1959). A program to simulate outcomes for German national and state elections under various alternative voting schemes has been developed by Carol Cassidy (1972). An interesting simulation which compares outcomes for many of the voting schemes in common use within small groups is Fishburn (1973). Of course, all such simulations fall prey to the danger of misrepresenting the consequences of changes in election systems, because of the failure of ceteris paribus assumptions. For example we would expect that a shift from a plurality to a list voting scheme would affect the numbers of candidacies offered and the campaign resource allocation patterns even for the existing parties (and thus the probable outcomes), even leaving aside the question of the long-run impact of the shift on the number of parties and the nature of party cleavages. For some additional relevant points on this issue see Sartori (1968).

IV. The Single Transferable Vote

Thomas Hare's method of the single transferable vote, although not the first scheme of proportional representation to be proposed (see Friedrich, 1968, p. 186) and although not in common use today (with the important exception, of course, of the Republic of Ireland and some parts of what was once the British Empire, (see Lakeman, 1974, pp. 278—280) remains of considerable importance because it is, by general consent, the "fairest" of all "PR" systems and has had (and continues to have) a number of distinguished advocates[32]). With the single transferable vote, in effect, each voter individually is able to choose his own constituency (i. e., his own representative) in accordance with his personal preference.

> "So many people who sleep inside an arbitrary line on the map — that is not the sort of constituency that ought to be condensed into a spokesman. It should be so many people who want the same spokesman. Each quota should be unanimous." (Hoag and Hallett, 1926, p. 4.)

Mill (1861) in order to maximize this aspect of the Hare System, advocated that the balloting for it be nationwide, thus maximizing the number of options open to each voter of individuals by whom he might choose to be represented.

Walter Bagehot (cited in Friedrich, 1968, p. 288), no fan of the Hare system, nontheless could see its attractions.

> "Under the compulsory form of constituency, the votes of the minorities are thrown away. In the city of London now, there are many Tories, but all the members are Whigs; every London Tory, therefore, is by law and principle misrepresented; his city sends to Parliament not the member whom he wished to have but the member he wished *not* to have. But upon the voluntary system the London Tories who are far more than 1,000 in number may combine; they may make a constituency and return a member. In many existing constituencies the disfranchisement of the minorities is hopeless and chronic." "Again this plan gets rid of all our difficulties as to the size of constituencies." "Again, the admirers of a great many could make a worthy constituency for him."

[32]) John Stuart Mill (1861) referred to the Hare System as one of the "very greatest improvements yet made in the theory and practice of government". The Proportional Representation Society in England, now the Electoral Reform Society (6 Chancel Street, London S. E. 1) of which Miss Enid Lakeman is the present Executive Director, has lobbied for over 100 years on behalf of the Hare System. (See eg. Lubbock 1968; Ross, 1951; Electoral Reform Society, 1965.)

Bagehot's view (echoed by Friedrich, 1968, pp. 290—292) was that these desirable features of the single transferable vote were clearly outweighed by what in his view was its inevitable destabilizing impact upon parliamentary government.

> "(U)pon the plan suggested, the House would be made up by party politicians selected by a party committee and pledged to party violence and of characteristic, and therefore immoderate representatives, for every 'ism' in England. Instead of a deliberate assembly of moderate and judicious men, we should have a various compound of all sorts of violence." (Bagehot, cited in Friedrich, 1968, p. 289.)

John Stuart Mill saw the Hare system as a check on the ascendancy of the numerical majority. Carl Friedrich (1968, p. 291), on the other hand, asserts that

> "Why should the problem of what is just to a minority be given precedence over what is just to the majority? Admittedly the majority wants action. Such action is, through proportional representation, being delayed or altogether prevented. What is the justice of that?"

Clearly, there are conflicting democratic norms here, and it is not the purpose of this paper to try to balance/resolve them. As far as we are aware, the only recent work on the mathematical properties of the Hare system has been done by Duncan Black (1969) and it is not of a nature to be easily summarized. Let us, therefore, simply note that for the Hare system in a district with m seats, $T_E = \frac{1}{m+1}$ (= 1 Droop quota), while no expression for T_R, purely in terms of first ballot preferences, seems possible[33]). In large part because of the efforts of the British Electoral Reform Society, there has, however, been a very close empirical scrutiny of the outcomes of STV elections, particularly as compared to those using simple plurality. (See e. g., Ross, 1959; Lakeman, 1974; and issues of *Representation*, the journal of the Electoral Reform Society.)

V. The Alternative Vote

The alternative vote (used in Australia) is sometimes put forward as a form of proportional representation. This is somewhat misleading. In terms of the seats/votes relationship, the alternative vote is a majoritarian

[33]) For a national legislature of r equally sized constituencies of m seats each, the national threshhold of exclusion is, of course, simply $\frac{m}{(m+1)\,r}$.

system and should behave much like plurality, except that for $n \geqslant 3$ we may show that, given certain plausible assumptions, minor "centrist"[34]) parties should be more often represented by the alternative vote than by plurality.

Consider a three party system with single member districts. Assume that the parties are seen by all voters as aligned on a left-right ideological continuum, that there is general rough agreement as to each's position on the continuum, and that each voter's preferences for each of the parties are determined by the distance between the voter's position in the ideological space and that of the party. Thus, for example, if the perceived alignment on the line ps were, i. e., $\dfrac{\;a\quad b\quad c\;}{p\;\; q\;\; t\;\; r\;\; s}$, with q as the midpoint of $\underline{ab}$, r of $\underline{bc}$, and t of $\underline{ac}$, then voters in the segment $\underline{pq}$ would have the preference ordering $\overline{abc}$, voters in the segment $\underline{qt}$ would have preferences bac, voters in the segment $\underline{tr}$ would have preferences bca, and voters in the segment $\underline{rs}$ would have preferences cba. Set $\underline{ps} = 1$. If we assume a rectangular distribution, then in a plurality system with sincere voting, $v_a = \underline{pq}$, $v_b = \underline{qr}$, and $v_c = \underline{rs}$.

Let us assume that $v_a > v_b$ or $v_c > v_b$ and that $\frac{1}{2} > v_a$, $\frac{1}{2} > v_c$; i. e., let us assume no party has a first ballot absolute majority. In a plurality system, party b will go unrepresented, despite the fact that if $\underline{pq} + \underline{qr} > \underline{rs}$ and $\underline{qr} + \underline{rs} > \underline{pq}$, then party b could receive an absolute majority in a *paired* plurality competition against *either* of the other parties alone, and this condition (under our assumptions) must *always* be met. Some simple algebra is sufficient to show that if $\underline{qr} > 0$, then party b will receive a majority in paired contest against each of the other parties. More generally, for any number of parties, if preference schedules are determined by the party's location in ideological space relative to that of the voter's ideal point, then there always exists a "centrist" party which can receive a majority against *each* of the other parties in pairwise competition, but which may nonetheless, be denied representation in a plurality system because of its lack of first ballot support[35]). On the other hand, with the alternative vote, for party b to go unrepresented, it must be the case that both $v_a > v_b$ *and* $v_c > v_b$. If party b survives to the second stage of the balloting, it will receive a majority.

We may extend this analysis by looking at thresholds of representation and exclusion. The threshold of exclusion for party b in the plurality system is, as we found before, $\frac{1}{2}$, but the threshold of exclusion for party b in a system using the alternative vote is only $\frac{1}{3}$ (it beats out at least one other party on the first round). Moreover, the threshold of representation for party b under the plurality system is $\frac{1}{3}$, but under the alternative vote it is only $\frac{1}{4}$ (the major party gets 50-percent, the second party gets 25-percent).

In general, for an n-party single member district system, the "centrist" party (i. e., the party whose candidate can receive a majority in paired contest against the candidates of each of the other parties can receive representation if it has as little as $\frac{1}{2(n-1)}$ percent of the first choice ballots (as compared with $\frac{1}{n}$ percent for the plurality case[36]). The threshold of exclusion in the n-party case is always $\frac{1}{3}$. Thus, the alter-

[34]) We are using here the term "centrist" in a quite specialized sense. By a centrist party we mean a party which, were it to be only a two party race, would receive a majority of the votes cast against any opponent. In other words, we are using "centrist" party as equivalent to "Condorcet choice" (See Black [1958] and Grofman [1969] for an etymology of this terminology). With this specialized meaning of "centrist" there may be no "centrist" party. Thus eg. Australia has no centrist party in our sense of the term. (I'm indebted to Professors Loveday and Sartori for calling this point to my attention.) Professor Sartori in his comments on this paper (personal communication) has pointed out what seem to me to be four additional ways in which one might talk of a centrist party. First, if we assume an ordinal left-right line-up, we may talk of a party as centrist if and only if there are parties to both its right and its left. If there are a proliferation of extremist parties parties on both sides of the spectrum this, of course, can be a quite inclusive definition. A second meaning of centrist comes if we assume that the left-right scale is interval (i. e., has a clearly defined distance metric) so that it makes sense to speak of those parties which are in the center of the spaces (in say, least squares terms) as being centrist. A third meaning of centrist comes if we are prepared to make the still more restrictive assumption that the left-right scale is a ratio scale. In this case there is a well-defined zero point, the region around which may be taken to be the center of the space. Finally, we may speak of a centrist party as one which, if the parties to either side of it were eliminated from electoral competition, would be chosen by voters of those parties in preference to abstention. I believe this notion of electoral substitutability is an important one for an understanding of multi-party dynamics. For an elaboration of it see Sartori (1975).

[35]) The condition we have imposed on voters preferences may readily be shown to be almost identical to Black's criteria of "single-peakedness". (See Grofman, 1969.) The proof for single-peaked preferences is given by Black (1958, Chapter 2).

[36]) We may not take $1/2\,(n-1)$ as the value of T_R, since there are some circumstances under which a party could obtain representation with even a smaller percentage of first round support. We have not been able to find a general expression for T_R. It appears, however, that as n grows large, T_R for the alternative vote in a single-member district can be virtually zero if we are prepared to make far-fetched enough assumptions about the distribution of voters' ideal points.

native vote does not guarantee that the "true" majority choice (i. e., that candidate who can receive a majority in paired contest against each and every other candidate) will be selected, but, for $n > 2$, such a candidate will be selected under less restrictive conditions (as to distribution of party strengths) in an alternative vote smd system than in a plurality smd system[37]).

Rydon (1968, pp. 191—192) notes (although not using this terminology) that there have been cases where in a three party race a Condorcet choice would have been eliminated at the first stage of the balloting but to prevent this, another party has declined to contest the election, knowing that if it did contest, the victor would be the candidate of the party least palatable to it, because the Condorcet choice would not garner sufficient first ballot votes if its candidate were also in the race to siphon first preferences away from it, while its candidate would lose in the second round paired contest against its least palatable opponent. In our terminology, party a, say, would have the motivation to engage in this self-sacrifice if $\underline{pq} + \underline{qt} < \underline{tr} + \underline{rs}$ while $\underline{pq} + \underline{qt} + \underline{tr} > \underline{rs}$.

It is important, however, to point out that regardless of this theoretical superiority of the alternative vote to simple plurality, in practice the alternative vote, as it has operated in Australia, could, according to one authority, just as well have been plurality for all the difference it made. Rydon (1968, p. 194) points out that on the average, in the period 1937—1961, 83 percent of the elections were decided by an absolute first ballot majority and only 5,4 % yielded results different from that which would have been obtained had simple plurality been the rule. She further asserts (Rydon, p. 194) that "there is no evidence that preferential voting increases the total number of candidates or parties".

Because the majority of the elections in the period were two party contests, and no breakdown between 2 and n party contests is provided, we are unable to estimate from Rydon's data the extent to which the alternative vote changed outcomes in those contests where more than two parties competed. My very tentative reanalysis of her data suggests that roughly half the constituencies which are contested by more than two parties are decided by absolute majority on the first ballot. Thus, of those contests in which the alternative vote *could* make a difference, roughly 34 %, about $\frac{1}{6}$, were decided differently by the alternative vote than they would have been by simple plurality[38]).

[37]) As far as we are aware, the analysis here is the first time that something like the single-peakedness assumption for voter preferences has been applied to the analysis of outcomes under the alternative vote. However, a similar but less formal analysis of the alternative vote may be found in Rydon (1968).

[38]) The approximate correctness of this analysis was supported by ECPR panelists from Australian National University who provided me with citations to raw data sources against which it could be checked. Unfortunately, I have not as yet obtained copies of these references.

VI. Cumulative Voting

We have earlier shown Thresholds of Representation and Exclusion and the Index of Distortion for the cumulative voting scheme. We now wish to conduct an analysis (similar to that we did for the limited vote) of optimal candidacy strategies; i. e., how many candidates should each party put up in any given constituency if it wishes to maximize the expected number of its candidates selected.

If we look at the case where $n = 2$, it is easy to see that we have a zero-sum two person game (see Luce and Raiffa, 1957). The first person to realize this and to apply game theoretic notions to cumulative voting apparently was Glasser (1959). If one party wishes to minimax; i. e., *guarantee* the selection of ℓ directors independent of how many candidates the other party puts up — then it must look to the worst possible case (see Luce and Raiffa, 1957) and give its ℓth candidate more votes than the other party can possibly give its $(m + 1 - \ell)$th candidate. Hence, the first party must have a vote percentage v sufficient to satisfy the inequality

$$\frac{mv}{\ell} > \frac{m(1-v)}{m+1-\ell} \qquad (6)$$

In other words, the first party should choose the maximum ℓ for which the above inequality holds and instruct its voters to divide their votes equally among the ℓ candidates. Moreover, if the other party is seeking to maximize its guaranteed seat share (and thus to minimize the first party's seat share), a similar calculation holds for it, and we may readily show (with some simplifying assumptions about remainders) that, in order to elect ℓ candidates, a party must have greater than an $\frac{\ell}{m+1}$ share of the vote. Hence, for the cumulative vote, the threshold of representation for ℓ representatives is simply ℓ times T_R. (Recall, for the cumulative vote, $T_R = \frac{1}{m+1}$.) Clearly if a party seeks more representation than it is "entitled to", it may be penalized for its presumption by being denied its "fair share" of the seats.

Where $n > 2$ the analysis becomes more complex. The minimax strategy still makes sense as a prudential one and is optimal of one's opponents are acting in concert. However, if one's opponents are divided and do not run only $m + 1 - \ell$ candidates, the minimax strategy may not achieve the maximum representation possible given the opposition's "irrationality". Without attempting to examine all of the details of n-party elections for $n > 2$, some theoretical properties of such games can be noted.

> "First of all, some elections will be 'inessential'. In such cases the sum of the number of [seats] each faction can win with a minimax strategy equals the total number of [seats] up for election ... For this type of game, the minimax rule given [above] provides a solution in the same sense as it does for the two-person game.
> However, other elections may be 'essential'; that is, the sum of the number of [seats] each faction can win with their respective minimax strategies may be less than the number of [seats being filled] ... If we wish to describe the opportunities for coalitions between factions in essential games, n-person cumulative voting elections can be described by a characteristic function which shows the results of a group of factions combining and using a minimax strategy in concert." (Glasser, 1959, pp. 155—156, with some change in notation.)

We shall not pursue these points further except to note that where $n = 3$, there exists only one essential cumulative voting game, and in it we may show that any coalition of two parties can, by coordinating their strategies, guarantee the winning of exactly one extra seat for the coalition. (See Glasser, 1959, p. 156; Luce and Raiffa, 1957; Brams, 1975).

In general, and even where $n = 2$, the minimax strategy may turn out not to be optimal for a party if its information about the distribution of voting strengths between it and its opposition is not certain. Brams (1975, pp. 110—115) shows that there are circumstances in which it is preferable to run one candidate more than the number dictated by minimax considerations. This strategy of running $\ell + 1$ candidates [which Brams (1975, pp. 112—113) shows to be a "dominant" one in the game theoretic sense (see Luce and Raiffa, 1957)] has the potential advantage of earning an extra seat. On the other hand, as Brams (1975, p. 113) notes

> "(W)hen a group's estimated electoral support is based on incomplete information, it may be prudent for it to stick to a safe strategy even when its estimated strength indicates that it should pursue a dominant strategy. Either strategy is 'optimal' in the sense of assuring the group the same guaranteed minimum number of seats, but the advantage the dominant strategy offers for winning an extra seat may be more than offset by the risk that, should the group's actual electoral support fall below the dominant threshold, it would jeopardize the guaranteed minimum that a safe strategy ensures."

Cumulative voting is in common use in the U. S. for elections of corporate boards of directors (see Glasser, 1959 and references cited therein). Brams (1975, p. 116, derived from Glasser, 1959, pp. 154—155) provides an example of the hazards of a majority group's choosing a risky strategy which we find instructive enough to cite in full. It concerns a fight for control at the 1883 meeting of the board of directors of the Sharpsville Railroad Company, where six seats were at stake and the voting was by cumulative vote.

> "The majority group, with 53 percent of the votes, chose a strategy of dividing its votes equally among a full slate of six candidates, and the minority group, with 47 percent of the shares, chose a strategy of dividing its votes among a slate of four. The minority group easily captured the four directorships it contested with each of its candidates getting 11.8 percent of the vote to the 8.8 percent for each of the majority group's candidates. In fact, if the minority group had pursued the bolder strategy of contesting five directorships, it could have won all by giving 9.4 percent of all votes to each of its candidates.
> The minimax strategies, on the other hand, are for the majority group to pursue the dominant strategy of contesting four directorships, the minority group to pursue the safe strategy of contesting three. Adoption of these strategies by each group results in each group's winning three directorships. Thus, even the minority group took a chance in contesting four directorships, because if the majority group had followed their dominant (minimax) strategy, they could have won four of the six directorships against this nonoptimal strategy of the minority group.
> Why did the factions behave so foolishly in this election? Although it is hard to understand what motivated the majority faction to spread its votes thinly across all the directorships — and thereby in effect hand majority control of the board to the minority faction — the motivations of the minority faction are somewhat clearer: It apparently had a priori knowledge that the majority faction would act as it did. Despite its knowledge, however, it did not fully exploit this intelligence, as we have indicated, although it did capture majority control of the board of directors, which may have been its goal." (Brams, 1975, p. 116.)

In principle, it is possible for a party to choose an expected utility maximizing strategy by subjectively assigning probabilities to the number of candidates it expects its opponents to run and the vote share it believes its opponents command, but we shall not pursue this matter further here. (See Luce and Raiffa, 1957; Brams, 1975, p. 112, note 14 and references cited therein.)

One state in the U. S., Illinois, used cumulative voting for its lower house, the Illinois General Assembly. As described by Sawyer and McRae (1962, p. 837[39])

"(T)hree representatives are elected from each district, and each voter has three votes which he may distribute 3—0, 2—1, 1½—1½, or 1—1—1 among the candidates. Each party may nominate for the general election one, two, or three candidates."

The decision as to how many candidates to nominate is made in at least partial uncertainty as to the percentage of the vote that the other party will receive (for all practical purposes, Illinois is a two-party state) and often, though not always, of the number of candidates that the other party will nominate. We may show (Brams, 1975, p. 117) that under these circumstances optimal strategies are

in the 0 to 25 range run 1
(expect to see none elected)

in the 25 + to 40 — range run 1
(expect to see one elected)

in the 40 + to 50 — range run 2
(expect to see one elected)

in the 50 + to 60 — range run 2
(expect to see two elected)

in the 60 + to 75 — range run 3
(expect to see two elected)

in the 75 + to 100 range run 3
(expect to see three elected)

Sawyer and MacRae (1962, Table II, p. 341) have looked at the extent to which parties use their optimal strategies and Brams' (1975, Table 3. 3, p. 118) retabulation of their data shows[40]) that in only 39 percent of all elections did *both* parties adopt their minimax strategies, although in

[39]) See also Blair (1960, 1973); Kuklinski (1972).
[40]) Brams (1975, p. 118, note 21) points out errors by Sawyer and MacRae (1962, pp. 939—940) in specification of optimal strategies for the Illinois case in the 25 + to 40 —, 60 + to 75 — ranges. Brams (1975, p. 120, note 23) points out the corrections in their tabulations needed to compensate for these errors.

95 percent of the elections one party *or* the other did so. However, what is particularly "astonishing about the nomination strategies and election results is that in more than half of the elections (56 %), the minority/majority parties chose the strategy pair 'run 1/run 2' which can *never* be optimal for both parties whatever the outcome of the election". (Brams, 1975, p. 118, emphasis in original.) Only in the competitive run did a majority of the competitors simultaneously make use of their optimal (run 2/run 2) strategy. The majority plays it unbelievably safe. Even in cases where the majority party received over 60 % of the vote — cases where it could not hurt for it to have run 3 candidates — over 80 % of the time only two candidates were run. "Of these failures on the side of conservatism . . . the results of 12 % of all elections could have been changed (seats gained for the majority) if one party had changed its strategy to that of its minimax strategy." (Brams, 1975, p. 120.)

As a possible explanation for this peculiarly bashful electoral behavior on the part of the majority party, Sawyer and MacRae (1962, pp. 939—945) suggest that bipartisan agreements may be reached which in effect cede one seat in each district to the minority party. We may note that in competitive districts, no such agreements are needed since the optimal strategy pair for both parties (run 2/run 2) leads to an outcome in which the minority still gets one seat[41]). We must judge deviations from optimal strategy choice in the context of two points. First, given incomplete information as to vote strength, some parties who are retrospectively seen as having pursued a "suboptimal" strategy may have been pursuing a strategy which, at the time and in the light of their available information, appeared optimal. Secondly, politicians may be well aware of optimality considerations and yet choose not to act accordingly because of "understandings" with their opposite numbers in the other party. As Brams (1975, p. 120) puts it, "This is not to say that politicians act irrationally but rather that other considerations may displace the minimax logic".

Rosenthal (1974) has extended the Sawyer-MacRae (1962) analysis of cumulative voting to cover union elections under the modified list system which permits striking out of some candidates on the list.

"By asking some of its militants to vote for an opposing list in striking out the top names on the list a union aims to wipe out the competing union's leadership. All the unions have consequently perfected a counter strategy: the faithful strike out the bottom names on their own lists in order to protect the big brass and make sure that its reelection is virtually automatic." (Fremontier, 1971, p. 303—304, cited in Rosenthal, 1974, p. 1.)

[41]) Cf. Brams (1975, pp. 119—120) with whom we differ slightly on this point.

Rosenthal has shown that "such voting would appear to require that two unions both prefer a situation where one union increases its seats but loses its leadership", and, in general, "as long as the unions are free to vary the number of candidates", this type of voting is "unlikely to occur". (Rosenthal, 1974, p. 22.) We shall not, however, further pursue Rosenthal's highly technical treatment of this subject.

VII. Weighted Voting

Consider a weighted voting scheme which assigns to each of N constituency groupings (or representatives) the respective weights of $w_1, w_2, \ldots w_N$. Let us define some coalition S as *winning* wherever

$$\sum_{i \varepsilon S} w_i > q > \frac{1}{2} \sum_{i \varepsilon N} w_i \qquad (7)$$

Clearly, if $w_i = w_j$ for all i, j then each voting bloc has equal power. If $q = \frac{1}{2}$, we have a majoritarian scheme. If $q > \frac{1}{2}$, then we have a requirement of some special majority needed to pass legislation; q is called the *quota* of the game. (If q = 1, unanimity is required before the group can act.)

> "There are a large number of voting situations in which some individuals or blocks of voters effectively cast more ballots than others. Such weighted voting systems are found in governmental bodies ... in the U. S. Electoral College, in voting by stockholders in a corporation ..., as well as when strictly disciplined political parties vote as a single bloc." (Lucas, 1974, p. 1.)

It is sometimes argued that by adjusting weights in a weighted voting scheme, inequalities resulting from such things as constituency sizes can be compensated for. However, the principal thrust of recent formal work on weighted voting schemes has been to show that the distribution of "power" (i. e., the ability to influence the outcomes of a group's decisions) within a group using weighted voting bears no *simple* relationship to the relative weights assigned to the group's members. Power in a group is not simply a direct function of one's strength as measured by number of votes; thus, for example, giving representatives weights equal to the relative sizes of the constituencies does not (as we may show) give all voters equal representation: "Simple additive or division arguments are not sufficient ... more complicated relations are necessary to understand the real distribution of power." (Lucas, 1974, p. 2.)

Let us define a coalition in a weighted voters situation as *minimal winning* when the deletion of *any* member of it results in the coalition no longer being winning. The Nassau County (New York) Board of Supervisors has, for some time, used a weighted voting scheme with six representatives assigned weights proportional to the size of the constituencies which they represent. In 1958, this resulted in weights of 9, 9, 7, 3, 1, and 1, respectively (Banzhaf, 1965). It is easy to see that the last three representatives, in reality, have no power; they can never be members of a minimal winning coalition[42]). Moreover, the third representative, although only having 7 votes, in reality is as powerful as the representatives with 9 votes, since a coalition of any 2 of these 3 representatives is winning!

An even more graphic example can be constructed. Consider a group of three representatives with weights 49, 49, and 1. The representative with a weight of 1 is as powerful as the other representatives since his single vote is as sufficient to turn a non-winning (single-member) coalition into a winning one as are the 49 votes of either of the other representatives.

It is clear from these examples that voting power expressed in some formal or functional sense is not directly proportional to the number of votes one casts. Scholars in this area have been concerned with developing an index which reflects

> "The importance of the individual in casting the deciding vote which will guarantee that some issue will carry. It should compare all the opportunities which each voter has to be a sort of critical swing-man in causing a desired outcome. This index should depend upon the number of players involved, on one's fraction of the total weight, and upon how the remainder of the weight is distributed." (Lucas, 1974, p. 10.)

The two indices, proposed to meet these desiderata, which have received the most theoretical attention as well as application to real-world situations (reapportionment disputes) are the Shapley-Shubik Index and the Banzhaf Index. (See Riker, 1964.)

Shapley and Shubik (1954) introduced an index which is a special application of a more general value concept introduced by Shapley (1953). A voter's Shapley-Shubik value is the a priori chance that he will be the last member added to turn a losing coalition into a winning one, where all coalitions are assumed to be equiprobable. The Shapley-Shubik Index for player i is given by

$$\Phi_i = \Sigma \; \frac{(S-1)!\,(n-S)!}{n!} \qquad (8)$$

[42]) In game theoretic parlance such reprensentatives are called *dummies*.

where the summation is taken over all winning coalitions S for which S — {i} is losing, where S is the number of members in the coalition S, and where n is the number of members in the group.

The other major index of voting power was introduced by Banzhaf (1965).

> "[Banzhaf] is a lawyer and much of his work has appeared in law journals; and his index, ever more so than the one above, has been used in arguments in various legal proceedings ... Banzhaf's index is also concerned with the fraction of possibilities in which a voter is in the crucial position of being able to change an outcome by switching his vote." (Lucas, 1974, p. 16.)

A player is said to be a *swing voter* for a given expression of preferences if his vote switch could change the outcome either from passage to defeat or vice versa. How often a player appears in such a swing position is taken by Banzhaf as the relative index of his power. We shall not offer a more precise definition of the Banzhaf Index here (see Lucas, 1974, p. 16; Brams, 1975, pp. 165—166). It is useful, however, to note (see Lucas, pp. 16—17) that, while the Shapley-Shubik Index uses permutations of the players and is concerned with the order in which winning coalitions are constructed, the Banzhaf Index employs combinations and does not look at the chronological order in which the winning coalitions were formed.

These indices can be used to provide relative power calculations for some well-known voting systems. For example:

> "The U. N. Security Council consist of the 'big five' ... who are permanent members (with) veto power; plus ten 'small' countries whose membership rotates. It takes nine votes, the 'big five' plus at least four others to carry an issue."

This is equivalent to weights of 7 for the permanent members and 1 for the rotating members, with a quota of 9 (see Lucas, 1974, p. 20).

> "A rotating member can be pivotal in a winning coalition S if and only if S contains exactly nine countries including the five permanent members. There are $\frac{9!}{3!\ 6!}$ such different S which contain i and the corresponding coefficient in the Shapley-Shubik formula for this 15 person game is $\frac{(9-1)!\ (15-9)!}{15!}$. The product of these two numbers times the power index $\Phi_s = .001863$ for any nonpermanent member. A member from the 'big five' has index $\Phi_b = \frac{(1 - 10\Phi_s)}{5} = .1963.$"
> Lucas, 1974, p. 20.)

This means that permanent members in the U. N. have, in Shapley-Shubik terms, 105 times the "power" of non-permanent members. (Cf. Barrett and Newcombe, 1968; Coleman, 1964; Schwodiauer, 1968; the latter two discussed in Brams, 1975, pp. 182—190.)

Let us now consider schemes which assign weights directly proportional to constituency size. It is possible to show that, in terms of the Banzhaf Index, this will involve overweighting the more populous areas by a factor proportional to the square root of the vote weight assigned. Thus, if a constituency with 4,000,000 inhabitants were given *twice* the weight of one with 1,000,000 inhabitants, the inhabitants in each constituency can be shown to have roughly identical Banzhaf indices. We shall not attempt to prove here this palpably nonintuitive but nonetheless mathematically sound result. (See Banzhaf, 1966; Lucas, pp. 32—34, 52—54.)

The Electoral College can be viewed as a 51-person weighted majority game between the states (plus the District of Columbia). Each state receives a number of electoral votes equal to the number of its representatives in the House and Senate. Thus, since all states get two electoral votes as a "bonus" we might think that the small states were overweighted. In terms of the Shapley-Shubik Index, this turns out not to be so. Calculations for the 1972 election (Lucas, 1974, p. 60) show that, actually, the large states are somewhat overweighted $\left(\frac{\Phi_i}{w_i} > 1\right)$ overall, however, the results are remarkably close to proportionality, i. e., the ratio $\frac{\Phi_i}{w_i}$ varies only from .9706 at minimum (Wyoming) to 1.0558 at maximum (California). An analogous analysis for the Banzhaf Index with much the same results is given in Banzhaf (1968). Note, however, that we are here using "states" as our actors. If we look at what Lucas (1974, p. 64) calls the "combined game" in which we view individual voters as the actors and seek to estimate their power in the electoral college as the product of the power *of* their state and their power *within* the state, rather different conclusions emerge. For example, in this combined game (see Lucas, 1974, p. 64) we find such interesting results as that voters in New York are three times as influential as voters in the small District of Columbia. In general, in the combined game, the "power" of big state residents will be enhanced. (See also Mann and Shapley, 1964 and Hinich and Ordeshook, 1974 whose sophisticated analyses offer some important caveats to the practical relevance of earlier findings.)

There has been much recent fervor in the U. S. to either abolish the Electoral College or to modify it with something other than a state-by-state, winner-take-all system. (See Longley and Braun, 1972, and references cited therein.) Calculations as to the impact of several such proposed changes are provided in Banzhaf (1968). (See also Hinich and Ordeshook, 1974.)

Clearly, the two power indices we have discussed fail to capture all of what is commonly meant by "power" (Cf. Riker, 1964); nonetheless, they can provide valuable insight into the nonobvious and often counter-intuitive properties of many voting schemes in common use[43]). (Cf. Brams, 1975, p. 190, who has singled out a number of power relationships which these indices help us to understand.)

These power indices may also provide insight into aspects of electoral dynamics. For example, Brams and Davis (1973) have shown, given certain simplifying assumptions, that a U. S. Presidential candidate seeking to maximize his expected electoral vote should allocate his campaign resources to each state in proportion to the 3/2's power of the number of electoral votes of the state, i. e., should disproportionally invest in the larger states. In a further paper, Brams and Davis (1974, pp. 125—126) have claimed to find empirical support for this 3/2's rule in the four most recent U. S. Presidential elections, although they find an over-investment of campaign time and money in the small states. The assumptions underlying this claim have, however, been challenged (Colantoni, Levesque and Ordeshook, 1975; Brams and Davis, 1975).

Conclusion

The central thrust of recent formal work on election systems is that life is considerably more complicated than simple distinctions between "PR" and "non-PR" systems would ever suggest. The proportionality of the seats-votes relationship varies i. a., with n, m, $\frac{r}{m}$, and the distributions of party strengths across constituencies (and in a quite complex manner); it is not simply a function of system type. Moreover, proportional weights do not necessarily result in proportional power. We hope to have demonstrated that recent formal work can contribute to our understanding of how given election systems are likely to operate under differing conditions.

As Carl Friedrich (1968, p. 301) has pointed out

> "It is difficult to predict . . . the exact effect of any particular system. In 1958, France's return to a single-member constituency with double ballot was expected to favor established politicians with strong local following and hinder the relatively unknown candidates from the New Gaullist party. But the result was a Gaullist landslide."

Certainly, this example suggests considerable room for improvement in our predictive accuracy with respect to the probable outcomes of any proposed manipulations of a nation's electoral laws[44]). Election systems do have important direct independent effects. For example, Rae (1967) has found that most European post WWII single-party parliamentary majorities were actually artifacts of the electoral system. However, we cannot look at election systems in isolation from prevailing patterns of social and ideological cleavages. Often, election systems reflect (more even than they shape) the prevailing patterns of party competition and cleavage and their effect is almost always outweighed in importance by shifts in those patterns[45]).

[43]) For an intriguing alternative but related analysis of power in a collectivity see Coleman (1971); reviewed in Brams (1975, pp. 184—187).

[44]) Giovanni Sartori (personal communication) has gathered data that strongly support the view that the single most important variable is district size — the larger the district size, the greater the number of contesting parties. This variable had been almost completely neglected in most traditional treatments of electoral systems.

[45]) For more on this point see Sartori (1968).

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Zusammenfassung

In dem vorliegenden Artikel werden Arbeiten über mathematische Wahlsystem-Modelle reflektiert und weitergeführt. Für die Analyse theoretischer Eigenschaften von Wahlsystemen schlägt der Autor vier Indizes und eine graphische Lösung vor: der Index der „maximalen Verzerrung", die Schwelle des „Ausschlusses" und der Repräsentation, der Index der Nicht-Repräsentation und die Sitz/Stimme-Parabel (maximum/minimum seat/vote curve). Diese werden benutzt, um die erwartete Ähnlichkeit der Ergebnisse und mögliche Verzerrungen durch verschiedene Wahlsysteme (einschließlich „first-past-the-post"-Mehrheit, d'Hondt'scher Regel, St. Lague System und modifiziertem St. Lague System, kumulieren, bloc vote und beschränkter Stimmgebung) zu überprüfen. Große Aufmerksamkeit wird den Beziehungen zwischen Sitzen und Stimmen bei verschiedenen Wahlsystemen gewidmet, so insbesondere dem sogenannten „cube law" zwischen Sitz und Stimme für Systeme mit zwei Parteien und einfacher Mehrheit in einem Wahlkreis. Ebenso werden Machtgleichgewicht und strategische Überlegungen für gewichtete und kumulierende Stimmsysteme angeschnitten und schließlich die speziellen Eigenschaften alternativen Wählens und das Hare-System diskutiert. Ziel der Untersuchung ist die Erörterung des Beitrags, den neuere formale und mathematische Modelle zur Lösung der in der Literatur über vergleichende Regierungslehre und Parteien aufgeworfenen, ziemlich „alten" Fragen geleistet haben.

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