Necessary and Sufficient Conditions for a Majority Winner in n*-Dimensional Spatial Voting Games: An Intuitive Geometric Approach**

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Necessary and sufficient conditions for there to be a majority winner in n-dimensional spatial voting games are well known, but they are customarily stated in symbolic terms in a fashion which is virtually incomprehensible to those without some reasonable degree of mathematical training, and the proofs of the basic results are even less accessible to the nonmathematically sophisticated reader. We offer proofs of the key results in this area restricted to the important case where voter preferences are a simple function of distance, that is, where, in a choice between any two alternatives, voters prefer the alternative that is closer to their ideal outcome. Our proofs, unlike those customary in the literature, can be understood by anyone who can remember high school geometry. The nature of our proofs is such as to show how the basic n-dimensional results, including the Plott (1967) conditions, the Kramer (1973) sequential voting theorem, the McKelvey (1976, 1979) agenda manipulation result, the Shepsle (1979) germaneness restriction result, and the McCubbins and Schwartz (1985) budget constraint result, can all be derived as reasonably straightforward extensions of Duncan Black's famous median voter result in the one-dimensional case.

William Riker (1961, 1980, 1982), among others, has called attention to the dismaying theorems in the social choice literature that show that it is extremely unlikely that there is an alternative that is a clear majority winner. Those results have been taken to imply that those in control of agendas can manipulate outcomes rather like a magician making a volunteer pick a particular card from the deck (cf. Plott and Levine, 1978). While these results (e.g., Plott, 1967; Davis, DeGroot, and Hinich, 1972; McKelvey, 1976) are justly celebrated, they are also not well understood by most political scientists, who regard them as arcana whose details are not accessible to those without considerable advanced mathematical training.

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This essay originated when one of us was trying to state for the other, in a way that was relatively easy to understand and to picture, the conditions necessary and sufficient for there to be a majority winner, that is, an alternative that is preferred by a majority of voters to each and every other feasible alternative. In the case where voters are choosing among alternatives that can be thought of as points in an n-dimensional space, such conditions are well known, but results are customarily stated for the most abstract and general case. Moreover, the statement is either entirely in mathematical symbols or, if paraphrased in English, is expressed in terms (e.g., contract loci, differentiable utility functions, separating hyperplanes) that are not (to put it mildly) intuitively easy to grasp without a good deal of mathematical background.[1]

In the course of our conversation, we came to realize that if we restricted ourselves to the simple but important case where voters have an "ideal point" in the space and rank ordered alternatives by how close they are to that ideal point,[2] then it might be possible to state the necessary and sufficient conditions for existence of a majority winner in a way that was easy to picture. Also, we believed that one could provide proofs which would be understood by anyone who remembered high school geometry (and was prepared to put in some thought and effort). We also realized that our proof techniques could enable us to show the direct connections between Duncan Black's famous (1948, 1958) "median voter" result for the one-dimensional case and the n-dimensional "instability" results as well as various other results on agenda structuring.

This essay has five justifications. First and foremost, it is an attempt to "demystify" social choice theory for the nonspecialist reader and to make key results accessible. All of the theorems we give, for example, are stated in plain English. Moreover, the proofs can be understood by any reader willing to spend some time thinking them through. Second, it shows in an easy-to-picture way how agenda manipulating trajectories can be constructed. Third, it provides some useful lemmas which help to understand the basic dynamics of spatial voting games. Fourth, as a contribution to intellectual history, it highlights the way in which Black's (1948, 1958) "median voter" result laid the foundation for subsequent developments. This link to Black's result is much harder to see in the original abstract and general versions of many of the theorems we review, especially given the form in which the proofs of these theorems are customarily

[1] Even the admirably lucid discussion of equilibrium in spatial voting games in Riker (1982, 182–88) suffers from this problem, and he offers no proofs of any of the results he discusses—leaving the reader with only the original sources, which are all *highly* technical.

[2] A condition known in the social choice literature as "circular indifference curves."

presented. Fifth and finally, it provides in one place, and with a unifying presentation and common notation, eight of the most important results to date on majority rule in the spatial context: results taken from Plott (1967), Davis, DeGroot, and Hinich (1972), Kramer (1973), McKelvey and Wendell (1976), Rae and Taylor (1971), McKelvey (1976, 1979), Shepsle (1979), and McCubbins and Schwartz (1985).

The Key Results

The alternative, if there is one, which can beat every other alternative in a head-on-head contest is the obvious "majority" choice. Nonetheless, it is well known that (1) such a majority winner may not always exist and (2) voting schemes in common use (such as plurality) will not always pick the majority winner even when one exists (Black, 1958; Riker, 1982). To see that there may be no unique majority winner, we can consider the simplest three-voter ($\{A, B, C\}$) and three-alternative ($\{x, y, z\}$) example of the "paradox of cyclical majorities" (Condorcet, 1785; Black, 1958). Example 1 shows the distinct preference orderings of each of the three voters; for example, voter A has preferences x preferred to y preferred to z, as shown below.

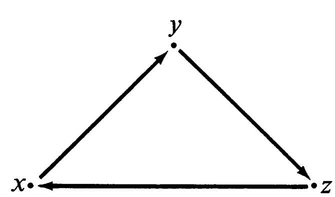
EXAMPLE 1: Voter	A	B	C
	x	y	z
	y	z	x
	z	x	y

In this example, voters A and C prefer x to z; voters A and B prefer y to z; yet voters B and C prefer z to x; that is, we have the cycle shown in Figure 1.

Black (1948, 1958) was the first to state an important condition sufficient to avoid a cycle and guarantee the existence of a majority winner. Before we state this and subsequent results, we state the key simplifying assumptions under which we shall be operating. We shall assume that each voter has one alternative that is his or her "ideal point" and that the

FIGURE 1

A Majority Preference Cycle among Three Alternatives



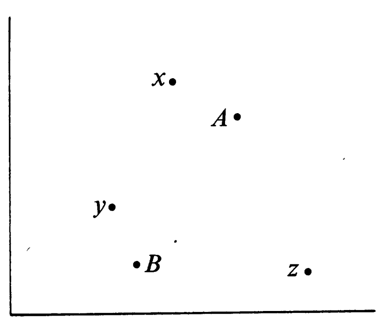
voter's preferences between any two other alternatives is determined by which of the two alternatives is closer to the voter's ideal point.[3] Throughout the rest of the paper we shall deal only with two dimensions or fewer, and we shall use an odd number of voters. The proofs we give generalize to higher dimensions, but we have not wanted to lose the intuitive aspects of the proofs by introducing such complications or the complications raised by considering an even number of voters and the possibility of ties. We shall also assume that voters always vote for the alternative that they prefer. We shall designate the alternative corresponding to a voter ideal point with a capital letter; all other alternatives will be designated with lowercase letters. We can illustrate the basic ideas with the simple two-dimensional example shown in Figure 2.

In this example, voter A prefers x to y because x is closer to voter A's ideal point. Similarly, voter A prefers y to z and also x to z. In spatial voting games under the given assumptions, every voter will have a transitive preference ordering (i.e., if xPy and yPz, then xPz, where P is used to denote preference); for example, voter B prefers y to z to x. Nonetheless, the group majority preference need not be transitive. For the three voters and the three alternatives in Figure 2, xPy (by voters A and C), yPz (by voters A and B), yet zPx (by voters C and B); that is, we have a cycle.

Unless otherwise stated we assume that every point in the space is an alternative that might be chosen. While in general the alternatives in

FIGURE 2

A Spatial Array of Three Voters $\{A, B, C\}$ and Three Alternatives $\{x, y. z\}$ Which Generates a Majority Preference Cycle



[3] Note that we are also implicitly assuming that voters' ideal points can be given a spatial interpretation and that the perceived spatial coordinates of any given alternative are identical for all voters, which means that there can be no assimilation or contrast effects or other forms of perceptual bias on the part of voters.

a spatial voting game need not be transitively ordered, there will always be a transitive majority ordering if all the voter ideal points lie on a single dimension, that is, on a straight line.

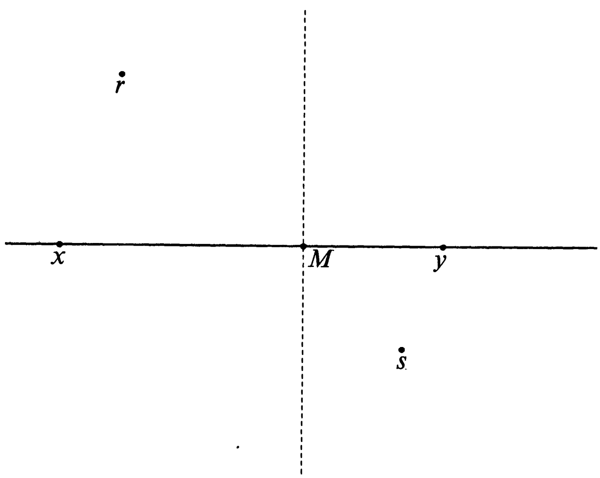
THEOREM 1 (Black, 1958): If the ideal points of all voters lie on a straight line, then there is a majority winner among the alternatives on that line, and that winner is the alternative corresponding to the ideal point of the median voter.[4]

PROOF: Consider the line shown in Figure 3. We have shown the ideal point of only the median voter.

Let *M* be the alternative corresponding to the ideal point of the median voter, *M;* that is, at least half the ideal points lie at or to the left of *M,* and at least half the ideal points lie at or to the right of *M.* Clearly, *M* will beat any point, *x,* on the line to the left of it, since voter *M* and

FIGURE 3

A Spatial Array of Voter Ideal Points along the Line with *M* as the Median Voter Ideal Point (*x, y, r, s* Are Some Alternatives in the Space)



[4]Black (1958) states the theorem in terms of single-peaked preferences. If all voters' ideal points lie on a straight line, then under the assumption given earlier about preference being determined by distance from the ideal point, voter preference orderings will be single peaked. As we present additional results, it will become clear why we have chosen to present the theorem in this form—a form which is weaker than the theorem actually proved by Black.

all the voters to the right will prefer M to x. But similarly, M will beat any point, y, on the line to its right. QED.

We can strengthen Theorem 1 to apply to the two-dimensional case as follows.

THEOREM 1′: If the ideal points of all voters lie on a straight line, then there is a majority winner among all alternatives in the space and that winner is the alternative corresponding to the ideal point of the median voter.

PROOF: Similar to proof of Theorem 1. We need only to look at points that are off the line and show that M beats them. Consider any alternative, r, to the left of M, but off the line. Again, all voters with ideal points on the line to the right of M prefer M to r and to any other point to the left of M. See Figure 2. Similarly, for s, any point to the right of M. QED.

The requirement that voter ideal points fall on a straight line (Theorem 1′) is a sufficient but not a necessary condition for there to be a majority winner. Plott (1967) established a more general sufficiency condition for multidimensional spatial voting games to have a majority winner. To prove his result and other subsequent results, we shall first prove some simple but important lemmas.

DEFINITION: The *projection* of a point on a line is the point which is the intersection with the line of a perpendicular line dropped from the point to the line. Figure 4 shows the projection, denoted a', of the point A corresponding to the ideal point of voter A.

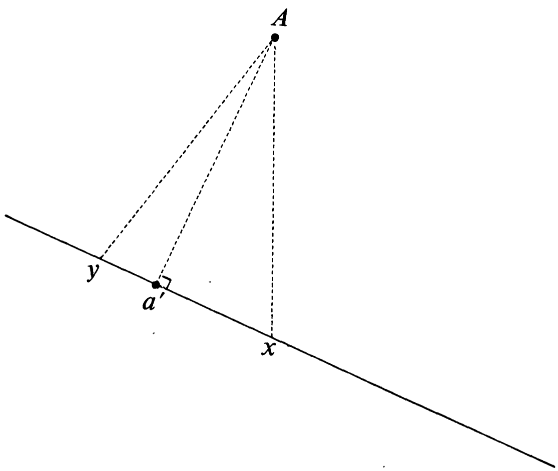
LEMMA 1: For any line each voter prefers points closer to the projection of his or her ideal point on that line to any other point on the line and prefers the other points in that line in order of their closeness to that projection.

PROOF: See Figure 4. By the Pythagorean theorem, for a right triangle, the hypotenuse must be the longest side, but then the distance from Voter A's ideal point to his projection on the line must be less than his distance to any other point on the line. The distance from an individual point A to any point x on some line is given by the Pythagorean theorem as d^2+f^2 where d is the distance from the voter ideal point to the projection on the line and f is the distance along the line from A's projection on the line to the point x. Thus, the smaller the distance, f, along the line, the closer is x to A and, thus, the more preferred. QED.

LEMMA 2: For any line the alternative that is the median projection of the voter ideal points onto that line is preferred by a majority of the voters to any other alternative on that line.

FIGURE 4

The Projection of a Voter Ideal Point, A, on a Line



PROOF: From Theorem 1, if all of the points were located on a line, then the majority winner would be the median ideal point. From Lemma 1, the preferences of any voter among alternatives on that line are given by distances from the voter's projection onto the line. Hence, the preferences of all voters among points on the line are as if the ideal point of each was at the voter's projection onto the line. Thus, the majority winner is the median projection on the line. QED.

LEMMA 3: If all voter ideal points fall on a single line, then the projection of the ideal point of the median voter onto any other line gives a point that is majority preferred to every point on that new line.

PROOF: The projection of points onto the new line will be in the same order as that of the points on the original line. Thus, the median point on the second line will be the projection of the median ideal point on the first line. QED.

Before we proceed to state Plott's (1967) sufficient condition for there to be a majority winner, we shall first state and prove four other interesting results about what happens when voters act as if their choices were confined to a line. Under our simplifying assumptions, these results can be readily derived from Theorem 1 and Lemmas 1 through 3. The first three results have been previously derived; the last of these (Theorem 5) is new, to the best of our knowledge, although it, too, follows directly from Theorem 1 and our subsequent lemmas.

THEOREM 2 (Shepsle, 1979; see also Shepsle and Weingast, 1981): If, in some multidimensional issue space, each bill (i.e., a committee or legislative proposal) is confined to a single-issue dimension and the only alternatives that can be considered are those germane to a bill (i.e., alternatives which lie on that dimension), then the position of the median voter on that issue dimension (i.e., the median voter projection) will be a majority winner vis-à-vis all other admissible alternatives.[5]

PROOF: Since the issue dimension can be treated as a line, the theorem follows directly from Lemma 2. The position of the median voter will be determined by the median projection of voter ideal points on the issue dimension. QED.

THEOREM 3 (Black and Newing, 1951; cf. Kramer, 1973): If there are n-issue dimensions in some n-dimensional issue space and if voters choose among alternatives confined to one issue dimension at a time, then the outcome will be independent of the order in which the issue dimensions are considered and will be the point that is at the median of each separate issue dimension.[6]

PROOF: For each dimension, by Lemma 2, the median voter projection on that dimension will be chosen. Thus, the overall result will be the multidimensional median. QED.

Of course, different sets of (orthogonal) issue dimensions will give rise to a different set of median points (see Feld and Grofman, 1985).

THEOREM 4 (McCubbins and Schwartz, 1985): If in a two-dimensional issue space, for example, one where the two dimensions are guns and butter, a linear budget constraint is imposed (i.e., total spending must not exceed some fixed amount, and each unit of guns costs some given unit price and similarly for each unit of butter), then the outcome will be the median projection of voter ideal points onto the budget line.

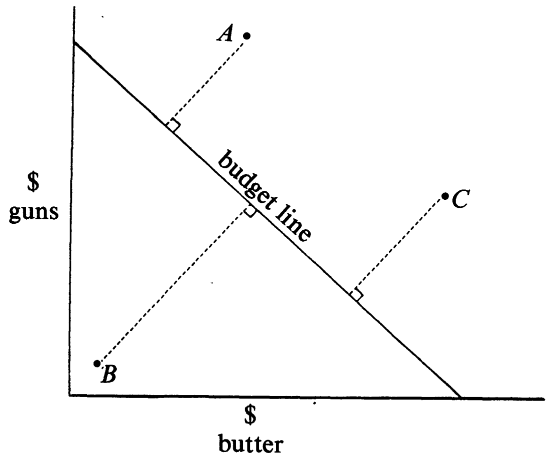
PROOF: See Figure 5.

[5] The Shepsle and Weingast (1981) paper also considers the consequences of sophisticated versus sincere voting strategies, but such issues are distinct from those that we shall pursue in this review essay.

[6] The actual central result in Kramer (1973) is stronger than the one we have stated and deals with sophisticated voting, but it too can be derived from Theorem 1′ and Lemmas 1 through 3 if we were to provide supplemental definitions such as one giving the meaning of the term sophisticated voting. To do so, however, would set us into issues beyond the intended scope of this paper.

FIGURE 5

The Choice among Alternatives along a Budget Line



The 45-degree budget line shown in Figure 5 reflects an equal unit cost of guns and butter, but any slope would have been possible. Points in the budget line show the combinations of guns and butter that can be bought subject to the budget constraint. Illustrative projections of voter ideal points onto the budget line are also shown. Clearly, by Lemma 2, the median projection onto the budget line is the majority winner.[7] QED.

The importance of the McCubbins and Schwartz (1985) result that there can be a majority winner if there is a budget line to which outcomes are confined, although only applying in two-dimensional space, nonetheless shows how restrictions on the set of feasible alternatives can affect the stability of majority rule decision making.

DEFINITION (Axelrod, 1970): A coalition is connected with respect to a given linear ordering if, whenever two voters are in the same coalition, so too are all voters who lie on the ordering between them.

THEOREM 5: If alternatives are limited to a single dimension (a line), then the only feasible coalitions are those coalitions that are

[7] This theorem holds only for two-dimensional issue spaces, since imposing a budget constraint reduces the dimensionality of the choice space by one. It can also be proved for a convex budget constraint. The McCubbins and Schwartz (1985) article is also of importance because it contains a perceptive discussion of what sorts of assumptions about the shape of voter utility functions make good empirical sense in various decision contexts.

connected with respect to the ordering of the projections of voter ideal points on that line.

PROOF: Consider a choice between any two alternatives on the line, say x and y. Consider any pair of voters, V_1 and V_2, who prefer x to y. If voter V is between V_1 and V_2, that voter must also prefer x to y, since the set of voters who prefer x to y are those who are to the right of the midpoint of the line segment between x and y. Thus, V_1 must be to the right of that point and so, too, must be V_2, and so must any voter V who lies on the line between V_1 and V_2. QED.

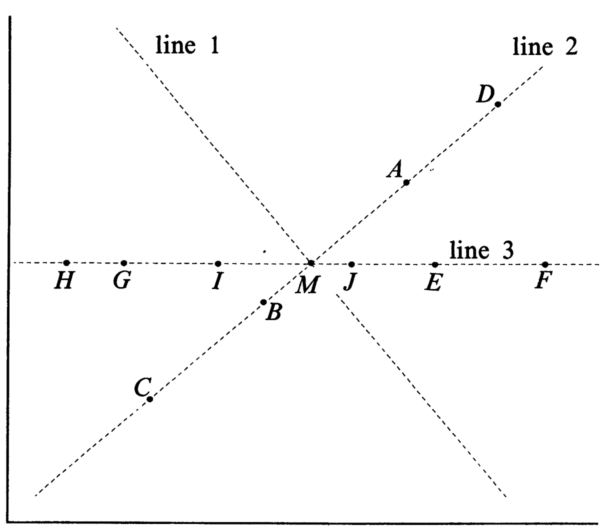
Now, we shall state and prove Plott's (1967) result.

THEOREM 6 (Plott, 1967): If there exists a voter's ideal point, M, such that every line passing through it has exactly as many voters' ideal points to the left of that point as to the right of that point, then the alternative M corresponding to that point is the majority winner (see Figure 6).

PROOF: Take any other alternative x. For any x there is a line passing through x and M. By the conditions of the theorem, all voters are located on some line through M. On each such line M is the median voter, since *every* line passing through M has exactly as many voters' ideal points to the left of that point as to the right of that point. By Lemma 3, for any given line, it follows that M is majority preferred

FIGURE 6

An Illustration of a Voting Game Satisfying the Plott Condition



by the voters whose ideal points fall on that line to any point, x, in the space. Since this is true for all lines, it follows that M will be majority preferred to any point, x, in the space. QED.

Theorem 6 provides sufficient conditions for a majority winner in terms of the actual location of voter ideal points. The Plott result, as stated, is sufficient but not necessary. Consider five voters, with two located inside a triangle formed by the other three. The location of the two internal voter ideal points is the Condorcet winner, but, technically, the Plott condition is not satisfied. (For a discussion of complications such as multiple voter ideal points at the same position, see Enelow and Hinich, 1983.) The next theorem will specify necessary and sufficient conditions in terms of the projections of ideal points rather than the ideal points themselves.

THEOREM 7 (McKelvey and Wendell, 1976): There exists a majority winner if and only if there exists a voter's ideal point, M, such that M is the median voter projection of voter ideal points onto every line which passes through M.

PROOF: Since M is by assumption the median of the voter projections on every line which passes through it, it is majority preferred to the points on all such lines, but all such lines include all points in the space. Thus, M is majority preferred to every point in the space as required.

To prove the only if clause, we need to show that if M is preferred to all points in the space it is the median projection of voter ideal points on all lines through it. Imagine the contrary; that is, imagine there is some line on which M is not the median. Call the median on that line x. From Lemma 2, the median on that line, x, must beat M, but that contradicts the assumption that M beats all points in the space. Therefore, a majority winner must be the median on every line that passes through it. QED.

To prove the next theorem we need a definition and two further simple lemmas.

DEFINITION: A *median line* is a line such that at least half the voter ideal points lie either on it or to the right of it and at least half the voter ideal points lie either on it or to the left of it.

LEMMA 4: If M is the median projection of voter ideal points onto a line, then the line drawn perpendicular to the line at that point is a median line.

PROOF: Voter ideal points to the left of that perpendicular line have projections onto the original line to the right of M; similarly voter ideal points to the right of that perpendicular line have projections onto the original line to the left of M. Since M is the median projection, this implies that half the voter ideal points must be on either side of the perpendicular line. QED.

LEMMA 5: If l is a median line then the point at which it intersects any line perpendicular to it is the median voter projection onto that line.

PROOF: Half the voter ideal points must be on either side of the median line. Thus, half the projections of these ideal points must be on either side of any line with which it makes a perpendicular intersection. QED.

Taking Lemma 4 and 5 together we have shown that a line l is a median line if and only if any line perpendicular to l has its median of projected voter ideal points at its intersection with l.

Now we can prove a useful version of the previous theorem.

THEOREM 8 (Davis, DeGroot, and Hinich, 1972; see also McKelvey and Wendell, 1976): There exists a majority winner if and only if there exists a voter's ideal point, M, such that every line passing through it is a median line. If so, the alternative, M, corresponding to that point will be a majority winner.

PROOF: From Lemma 4 and Lemma 5 it follows that any point that is the median of all voter projections on lines through it is on all median lines (i.e., all median lines intersect at that point), and conversely. Then Theorem 8 follows directly from Theorem 7. QED.

Theorem 8 says that, in two dimensions, a point M is the majority winner if and only if every median line passes through it.

We can now prove a necessary condition to complement the sufficient condition of Plott's theorem, Theorem 6, again in terms of voter ideal points rather than voter projections.

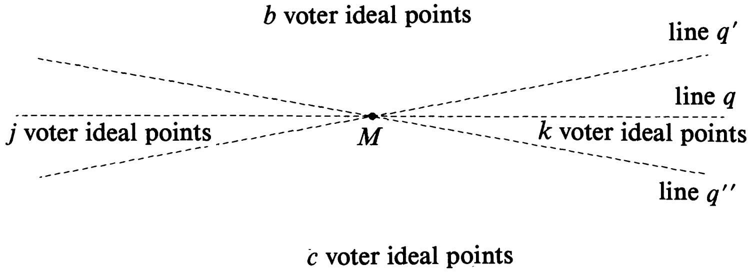
THEOREM 6′: If there is an alternative, M, that is the majority winner, then there exists a voter's ideal point M, corresponding to that alternative, such that M is the median voter ideal point on every line passing through it.

PROOF: To prove that the existence of a majority winner implies this version of the Plott condition, we provide a proof using Theorem 8. Suppose that there is some line, q, through M where there are j voter ideal points on one side of M and k voter ideal points on the other. We will show that j must equal k.

Suppose there are b voter ideal points on one side of the line, q, and c voter ideal points on the other side, as shown in Figure 7. Theorem 8 indicates that all lines passing through a majority winner, M, are median lines. Consequently, lines q' and q'', through M, barely on opposite

FIGURE 7

A Majority Winner, M, on the line q, with k Voter Ideal Points to Its Left on the Line and j Voter Ideal Points to Its Right on the Line



sides of q, must be median lines. For q' to be a median line, it can be seen that the number of voter ideal points on one side must be equal to the number on the other:

$$b + j = c + k.$$

Similarly, for q'' to be a median line, the number of voter ideal points on one side must be equal to the number on the other:

$$b + k = c + j.$$

Subtracting the second of these equations from the first yields:

$$j - k = k - j.$$

This can only be true where $j - k = k - j = 0$; therefore, $j = k$. QED.

Some Additional Results

DEFINITION: The *top cycle set* is the minimal set of points, such that no element of the set is beaten by any element outside the set. The top cycle set consists of points that are part of a cycle with every point that beats them and that collectively can beat any point in the space.

THEOREM 9 (McKelvey, 1976, 1979): In spatial voting games, if there is no majority winner, then there exists a path between any two points r and s such that r and s are a part of a cycle; that is, there exists x, y, etc., such rPx, $xP\_ \ldots \_Ps$ and similarly sPy, $yP\_ \ldots \_Pr$, that is, if there is no majority winner then every point can be made part of the top cycle.

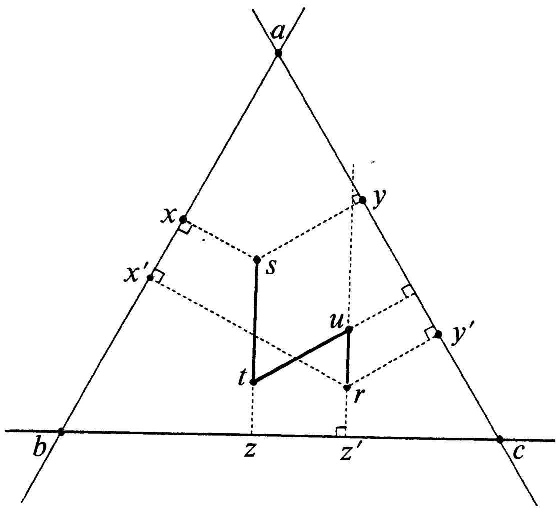
PROOF: In order to prove this result, we make use of theorems that we have previously proved. Specifically, if there is no majority winner, then there is no point at which all median lines intersect. We can take any two median lines and find their intersection and then find some other median line that does not intersect them at that same point (see Figure 7). The three median lines in Figure 8 form a triangle with vertices *a, b,* and *c* and extend beyond that triangle. We shall divide our discussion into consideration of points within the triangle and outside of it. Before we can complete the proof, four new useful lemmas are needed.

LEMMA 6: For any two points *r* and *s* within a triangle of median lines, there is a path *rPu, uP_ . . . . _Ps.*

PROOF OF LEMMA 6: In Figure 8 it is easily seen that any point that is directly closer to a median line than *s,* for example, *x,* is majority preferred to *s* because the median on that perpendicular line is on the median line. So *s* is beaten by *x* and *y* and *z.* Moreover *s* is beaten by the points on the line segments $\overline{xs}$, $\overline{zs}$, and $\overline{ys}$, which are closer to their respective edges of the triangle than is *s.* In like manner *r* is beaten by the points on the line segments $\overline{x'r}$, $\overline{z'r}$, and $\overline{y'r}$, which are closer to their respective edges of the triangle than is *r.* To construct a majority preference trajectory from *r* to *s,*

FIGURE 8

A Majority Path from *s* to *r* Points within a Triangle Defined by Three Median Lines



we move along lines perpendicular to the medians until we can create an intersection with one of the three lines $\overline{rx'}$, $\overline{ry'}$, or $\overline{rz'}$.

In order to make this clear an illustration is required. Begin at s. Move along $\overline{sz}$ until you hit some point t. Clearly tPs. Now move from t in a direction perpendicular to one of the median lines, say $\overline{ac}$. In so doing we intersect the line $\overline{rz'}$ at u. Clearly uPt and rPu. While we may have to make additional intermediate moves to create some majority trajectories, it is possible to get from any point in the interior of the triangle to any other in this fashion. Since we made no assumption as to whether rPs or sPr, by the same technique we can create a majority dominance trajectory between s and r rather than between r and s as above. We can move in any of three directions within the triangle and thus go from any point to any other point within the triangle. QED.

Lemma 6 provides the proof of the desired result for trajectories which are entirely within the "median triangle." We now need to show how to cope with exterior points.

LEMMA 7: For any two points on a line, the point that is closer to the median voter projection on that line is majority preferred to the other.

PROOF OF LEMMA 7: Voters will choose between two points on a line based upon which of the points is closer to their projection on the line (Lemma 1). All voters whose projections are on one side of the midpoint between the two points will vote for one, while all voters whose projections are on the other side will vote for the other. Whichever side of the midpoint has the median projection has more than half of the voter projections. QED.

LEMMA 8: For any point r outside the triangle made by median lines, there is a chain of indifference leading to a point, q, within the triangle, that is, rIx, $xI\_$, . . . , $\_Iq$, where q is inside the triangle and I denotes indifference.

PROOF OF LEMMA 8: As shown in Figure 9, if r is outside the triangle, it must be outside at least one of the lines bounding the triangle. If another point r' is determined to be equidistant from that line on the other side of the median line as shown in Figure 9, then by Lemma 7, the voters must be indifferent between r and the new point r'. Furthermore r' must be closer to the center of the circle inscribed in the triangle. It is clear that r' on the same side of the boundary line as the center of the circle is closer to the center of the circle than is the original r. Now if the new point r' is still outside the triangle, then it is outside some other boundary line. A new point r'' can be found equidistant from that line. By the previous argument, the new point is majority indifferent to r' and closer to the center of the inscribed circle.

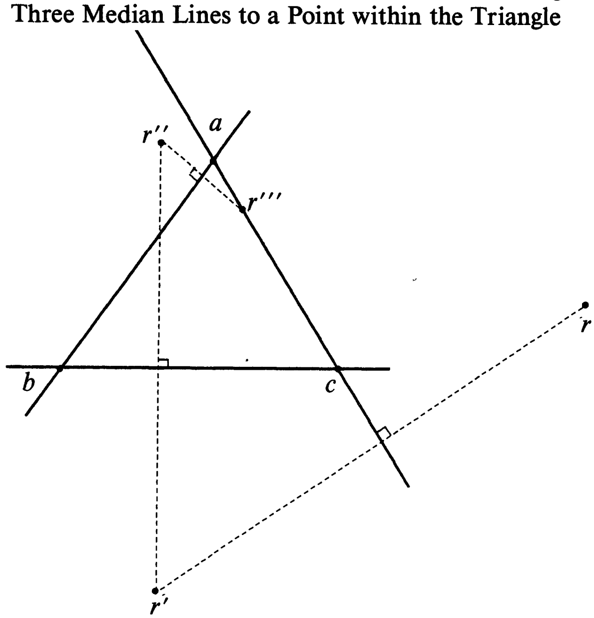
The process can be continued until the chain reaches a point inside the triangle. In the example given in Figure 9 this chain ends at r'''. QED.

LEMMA 9: For any point r outside a triangle made by median lines, there is a chain leading to within the triangle, that is, rPx, $xP\_$, . . . , $\_Pq$ where q is inside the triangle; and for any point s outside a triangle made by median lines, there is a chain leading from within the triangle: tPx, $xP\_$, . . . , $\_Ps$, where t is inside the triangle.

PROOF OF LEMMA 9: From the previous lemma, there is a chain of indifference. If the new points at each stage are made infinitesimally closer to the median line, then they are majority preferred at each stage as they become closer to the center of the inscribed circle. If the new points at each stage are made infinitesimally further from the median line, then they are majority inferior at each stage as they become closer to the center of the inscribed circle. QED.

FIGURE 9

A Path of Majority Indifference from a Point, r, outside a Triangle Defined by Three Median Lines to a Point within the Triangle



Now we can complete the Proof of Theorem 9: (a) If the two points r and s are within the triangle, then Lemma 6 is the proof; (b) if r is outside and s is inside, then Lemma 9 shows that there is a path into the triangle and Lemma 4 shows that there is a continuation of the path within the triangle; (c) if both r and s are outside the triangle, then Lemma 9 shows that there is a forward path from r into the triangle and a backward path from s into the triangle. Lemma 4 shows that there is a connecting path within the triangle. QED.

Theorem 9 has the disturbing implication that if there is no alternative that is majority preferred to every other alternative, then there is always some chain of alternatives that can move the group by a path of majority preference from any alternative, however popular, to any other alternative, however disliked.[8] This shows the extraordinary potential instability of majority vote procedures and the seemingly tremendous opportunities for agenda manipulation. However, the chain of alternatives that would be needed to move from a starting point near the "center" of the space to alternatives that are far from the "center" of the space will usually be quite long (Feld, Grofman, and Miller, 1985), so this result is not quite as dismaying as it first appears because restrictions on the number of agenda items will constrain the number of intermediate steps and thus make it unlikely that "far out" alternatives can become the majority choice.

Discussion

The results we have given provide, we hope, the basis for an intuitive understanding of many of the most important results about majority rule spatial voting games. They show that three basic results on equilibrium in n-dimensional voting games—the sufficiency condition discovered by Plott (1967), the necessary and sufficient conditions first stated by Davis, DeGroot, and Hinich (1972), and the agenda manipulation results of McKelvey (1976, 1979)—can each be proved with methods based on Black's median voter result for ideal points all located on a single line. In particular, only if a point is a median point in either the particular sense of Theorem 5 (a *median voter ideal point* on *every* line that passes through it) or the more general sense of Theorem 6 (a *median of the voter projections* on *every* line that passes through it) can that point be a majority winner. These direct connections between Black's (1948, 1958) basic result and subsequent theorems can readily be lost sight of.

In the multidimensional spatial context, necessary conditions for the existence of a majority winner are quite restrictive. To have a point be

[8] Note that some of the points in the top cycle set are *Pareto dominated;* i.e., there is at least one alternative that all voters prefer to it.

the median point in the projections onto *every* line that passes through it imposes a condition of near perfect symmetry on the distribution of voter ideal points. To require a germaneness constraint or one-issue-at-a-time decision making is also restrictive, as is the assumption of exactly two dimensions and a linear budget constraint or the assumption of unidimensionally connected coalitions.[9] However, there are reasons why social decision processes may still be relatively well behaved even in the absence of a majority winner or restrictions of choices to single dimensions. First, if we move from a deterministic to a stochastic framework, the *probable* outcomes of most reasonable agenda processes may be found in a small portion of the space (Ferejohn, McKelvey, and Packel, 1984; cf. McKelvey, 1986; Feld, Grofman, and Miller, 1985). Second, if we move to supramajoritarian quorum rules, at least in two or three dimensions a core is likely (McKelvey and Schofield, 1984; Schofield, forthcoming; Schofield, Grofman, and Feld, forthcoming). Third, once an outcome has been agreed upon it may prove very resistant to change because of various forms of transaction costs (see Grofman and Uhlaner, 1985). Fourth, even though there may not be a majority winner, there may nonetheless be strong centripetal pressures which move outcomes toward regions near the center of voter ideal points. In particular, there is a considerable "internal structure" to majority choice in the spatial context that severely constrains the feasibility of agenda manipulation (McKelvey, 1986; Feld, Grofman, and Miller, 1985; Feld et al., forthcoming; Grofman et al., forthcoming; Wuffle et al., forthcoming).

These new directions in spatial social choice theory are very recent ones, and it is too early to tell which will be of lasting significance.

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[9] Also, if voters use the city-block metric to evaluate alternatives, i.e., if for each voter the utility of any alternative is the sum of its utilities on each (orthogonal) dimension separately, then the point which is the median choice on each of the dimensions will be a majority winner (Rae and Taylor, 1971). This result follows straightforwardly from Lemma 2 and is very similar to Theorem 3 in the text. The Rae and Taylor (1971) result can also be derived as a special case of a result on decision making with separable preferences given in Kadane (1972).

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