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THE GEOMETRY OF MAJORITY RULE*

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ABSTRACT

We present some basic results concerning the spatial theory of voting in such a way that the theorems and their proofs should be accessible to a broad audience of political scientists. We do this by making the presentation essentially geometrical. We present the following results in particular: Plott's 'pairwise symmetry' condition for an unbeaten point; McKelvey's 'global cycling' theorem; Ferejohn, McKelvey and Packel's cardioid construction for establishing bounds on a 'win set'; and McKelvey's circular bound on the 'uncovered set' of points.

KEY WORDS • majority rule • spatial voting models

Most political scientists are probably aware of a line of research, within the general rubric of positive political theory, that is referred to as the spatial theory of voting. They may further be aware of the somewhat disconcerting nature of the fundamental results of this research, which are often referred to as the 'chaos theorems'. These results appear to imply that majority rule over an alternative space of two or more dimensions is disorderly and that, accordingly, political choice may be highly unstable or arbitrary. However, the details of these theorems – and their proofs in particular – probably remain beyond the understanding of almost all who are not specialists in the area, since they are stated and proved in highly abstract terms, using elaborate symbols and advanced mathematics.

The purpose of this essay is to present some basic results in the spatial theory of voting in such a way that the theorems and their proofs are as accessible as possible to a broad audience of political scientists. Our motivation is provided by the conviction that these theorems are highly relevant for political science and that they ought to be more widely understood. Accordingly, our method of presentation is essentially geometrical, and no mathematics beyond the high-school level is employed. However, while the mathematics is elementary, the logical deductions are necessarily intricate and cumulative; accordingly the essay will likely require close study on the part of readers who wish fully to grasp the arguments.

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We present the following theorems in particular: the 'pairwise symmetry condition' theorem for a 'majority rule equilibrium' due to Plott (1967); the 'global cycling' theorem due to McKelvey (1976, 1979); the cardioid construction for establishing bounds on a 'win set' due to Ferejohn et al., (1984); and the application by McKelvey (1986) of this construction to establish a circular bound on the 'uncovered set' of points. The substantive implication of our discussion is that the majority rule over a multidimensional space is typically more orderly than discussions of the 'chaos theorems' often suggest.

1. Overview

A spatial voting game has two elements: a multidimensional alternative space and a finite set of voters with preferences defined over this space. Each point in the space may be interpreted as a possible combination (on each of the several dimensions) of policies, programs or budgets, or as possible electoral platforms for political parties or candidates identifying them with such combinations of policies. Alternatively, the space may represent the several ideological dimensions (e.g. economic liberalism – conservatism, social liberalism – conservatism, etc.) in terms of which policies are commonly perceived and differently evaluated. In general, then, the space represents alternatives available for political choice, over which people have differing and more or less conflicting preferences. In particular, we suppose that each voter has preferences over all points in the space, i.e. given two points x and y, a voter prefers x to y, or prefers y to x, or is indifferent between the two.

Given voter preferences, a *majority preference relation* is generated between every pair of points. We say that x *beats* y if more voters prefer x to y than prefer y to x, that y *beats* x if the reverse is true, and that x *ties* y if the same number of voters prefer x to y as prefer y to x.[1]

It has become customary to call the set of points that beat x the *win set* of x and to designate it $W(x)$. A point that cannot be beaten under majority rule, i.e. a point x such that $W(x)$ is empty, is variously called a

1. A more stringent definition of majority preference is this: x beats y if more than half of all voters prefer x to y (rather than more than half of all voters who are not indifferent between x and y), y beats x if the reverse, and x and y tie otherwise. This may be called *absolute* majority rule, in contrast to *relative* majority rule as defined in the text. Absolute majority rule is often assumed in the voting theory literature, partly because it gives slightly 'cleaner' results (see footnotes 5, 11 and 12) and partly because it allows majority preference to be defined in terms of 'winning coalitions' in the sense of game theory. However, the definition of relative majority rule better matches ordinary usage and practice. Furthermore, in the spatial context, whether one uses the relative or absolute definition makes little difference, especially if the number of voters is odd. Exceptions are noted in the footnotes cited above.

'majority rule equilibrium', a 'Condorcet winner', or a majority rule 'core'. Here we call it simply an *unbeaten point*.

The existence of an unbeaten point would seem to be required for stable political choice, since, in the absence of such a point, whatever combination of policies momentarily prevails, there is some majority coalition of voters with both the collective power (given majoritarian institutions) and the common desire to upset that policy status quo and replace it with something else. Moreover, on some interpretations, the existence of such a point is necessary to fulfill the prescriptions of the normative theory of populist democracy (cf. Dahl, 1956). However, Plott's (1967) theorem and a number of related results indicate that such a point almost never exists in a space of two or more dimensions.

This is because majority preference may *cycle* – that is, it may be that x beats y, y beats z, and so forth, to some v such that v beats x.[2] If there is no unbeaten point, there must be at least one majority preference cycle, such that every point in the cycle is beaten by some other point in the cycle while every point outside the cycle is beaten by every point in the cycle. The smallest set of points each of which beats every point outside the set is called the *top cycle* set.

Plott's theorem left open the possibility that the top cycle set in a space of two or more dimensions might be a small subset of the space. This would imply that, even though political choice would almost never be fully stable, it might be 'approximately stable'; while majorities would always have the power and desire to upset the status quo of the moment, they could replace it only with some 'nearby' (i.e. only incrementally different) point, and – more to the point – given even an indefinitely long accumulation of such incremental changes, a political choice process driven by majority rule could wander over only a small portion of the space (as long as preferences remained constant). This hope was dashed, however, by McKelvey's 'global cycling theorem' (1976), which demonstrated that if, as Plott's theorem showed is almost always the case, majority rule fails at all, i.e. if there is no unbeaten point, it fails completely, i.e. the top cycle set encompasses the entire alternative space. Apparently 'anything can happen'.

More recent results, however, have suggested that majority rule has some deeper structure that guides and constrains many voting processes even in the face of all-encompassing cycles. The following definition is important. Point x *covers* point y if and only if x beats y, x beats everything y beats, and x beats or ties everything y ties. The *uncovered set* is the set of all points none of which is covered by any other point.

In a more recent paper, McKelvey (1986) has shown that several competitive political processes (electoral competition between power-oriented

2. Majority preference is *transitive* if, whenever x beats or ties y and y beats or ties z, x beats or ties z. Transitivity is sufficient, but (if ties occur) not necessary, to avoid majority preference cycles.

parties or candidates, an open agenda-formation process followed by sophisticated voting under standard amendment procedure, and 'cooperative' voting with coalition formation) are all driven into the uncovered set of points, and that – at least for 'Euclidean' voter preferences (which we define below) – this set is centrally located and may be relatively small.[3]

McKelvey's bounds on the uncovered set were derived from a construction developed earlier by Ferejohn et al. (1984) that established inner and outer bounds on the win set of an arbitrary point in a multidimensional space, again provided voter preferences are 'Euclidean'. Quite apart from its relevance to voting processes that produce outcome in the uncovered set, this construction allows us to gain a general understanding of the character of win sets in spatial voting games. It also implies that the majority preference relation is rather more orderly than the global cycling theorem may suggest.

2. Assumptions

For the purposes of this article, we make several simplifying assumptions.

First, we assume that all voters have 'Euclidean' (or 'Downsian', or 'Type I') preferences. This means that individual preference is based on simple Euclidean distance, i.e. each voter has an *ideal point* (point of highest preference) in the space and, in comparing any two points in the space, prefers the point closer to this ideal to the point more distant from it, and is indifferent between them if they are equidistant from the ideal. Since, in two dimensions, the locus of points equidistant from a fixed point is a circle, a voter's preferences relative to an arbitrary point x in a two-dimensional space can be represented by a circle (called an *indifference curve*) centered on the voter's ideal point and passing through x. Every point inside the circle is closer to the voter's ideal point than x is, so the voter prefers any such point to x. All points on the circle (including x) are equidistant from the voter's ideal point, so the voter is indifferent among all such points. And every point outside the circle is further from the voter's ideal point than x is, so the voter prefers x to any such point. While this assumption is restrictive, it provides a reasonable approximation to many important situations. In any event, we are in this respect following McKelvey (1976) and the relevant portions of Ferejohn et al. (1984) and McKelvey (1986).[4] Finally, we have the strong intuition that the general thrust of the results presented here extend to spatial voting games with more general preferences.

Second, for ease of exposition and the presentation of diagrammatic

3. McKelvey's analysis for the spatial case generally parallels an earlier analysis by Miller (1980), for a finite set of discrete alternatives over which voters have unrestricted preferences.

4. Plott (1967) stated his theorem for more general preferences, and McKelvey (1979) generalized his global cycling theorem.

examples, we focus on the case of a two-dimensional, rather than a general multidimensional, space. When standard assumptions (much more general than the Euclidean assumption made here) are made about voter preferences, the character of majority rule changes rather fundamentally as we move from one dimension to two. In particular, many years ago Black (1948) showed that an unbeaten point always exists in a one-dimensional space, namely the median voter ideal point. But, by Plott's theorem, an unbeaten point almost never exists in two (or more) dimensions. It is true that, when we move from two dimensions to three (with an odd number of voters, or from three dimensions to four with an even number of voters), there are further changes in the character of majority rule (cf. Schofield, 1982), but we do not here address these matters (which pertain to 'continuous' trajectories through the space). With respect to our present concerns, the two-dimensional case adequately illustrates general properties of majority rule in spatial voting games and, for the most part, our discussion generalizes straightforwardly to the multidimensional case. (Exceptions to this statement are indicated in footnotes.)

Finally, for analytical convenience, we deal only with the case in which the number of voters n is odd. The import of this restriction is indicated by Lemma 1 below. It should be noted that two points may tie even if n is odd, because an odd number of voters may be indifferent between them.

In sum, in specifying a spatial voting game, we have a two-dimensional alternative space and a finite odd number n of voters with ideal points distributed in the space. Since preferences are Euclidean, the location of ideal points determines all preferences, and x beats y if and only if, of all ideal points not equidistant from x and y, a majority are closer to x, and y beats x if and only if a majority are closer to y. Points x and y tie if and only if, of all ideal points not equidistant from x and y, exactly half are closer to x and half to y. Given these assumptions, our analysis can proceed on the basis of mathematics no more advanced than high-school plane geometry and trigonometry.

3. Median Lines

Any line L through a two-dimensional alternative space partitions the ideal points into three sets: those that lie on one side of L, those that lie on the other side of L, and those that lie on L. A *median line*, M, partitions the ideal points so that no more than half of them lie on either side of the line. It follows immediately that, if – as we assume throughout – the number of ideal points n is odd, any median line M must pass through at least one point, for otherwise there would be at least $(n+1)/2$ points on one or other side of M. It likewise follows that no two median lines M and M' can be parallel, for there must be at least $(n+1)/2$ points on and to either side of median line M and thus at least the same number strictly to one side of any other median line M' parallel to M.

Usually, an ideal point has an infinite number of median lines passing through it, and (with n odd) most median lines pass through just one point each. For some purposes, we need be concerned only with a finite subset of median lines that may be called *limiting* median lines, each of which passes through (at least) two ideal points. Figure 1, which shows a configuration of five ideal points ($x^1, \ldots, x^5$) displays one non-limiting median line M passing through ideal point x^1 and all limiting median lines through all five ideal points (each designated M_{ij} according to which pair of ideal points x^i and x^j it passes through). Note that any line passing through x^1 in the cone formed by M_{13} and M_{15} is a (non-limiting) median line, and likewise for the other ideal points.

We now present several elementary lemmas.

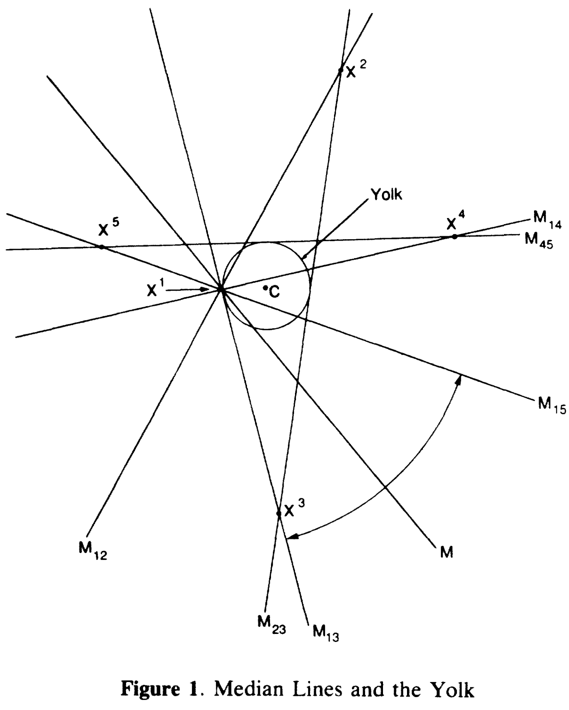

Figure 1. Median Lines and the Yolk

LEMMA 1. *Given any line L, there is some median line perpendicular to L. If n is odd, there is exactly one median line perpendicular to L.*

Proof. Erect a line perpendicular to L. Now shift this perpendicular line to the left or right until it is a median line. With n odd, only one line will do (because we cannot have parallel median lines). It will pass through one and, almost always, only one point.

The second sentence of Lemma 1 demonstrates the import of the assumption that the number of voters is odd. Henceforth, this assumption, like those restricting the analysis to two dimensions and to Euclidean preferences, should be taken as implicit in all lemmas and theorems.

LEMMA 2. *Given any two points x and y, if the median line M perpendicular to the line through x and y is closer to x than to y, then x beats y, and if M is closer to y, then y beats x. And x ties y only if M is equidistant from x and y.*

Proof. Given Euclidean preferences, every voter prefers x to y or y to x according to which is closer to his or her ideal point. The division of preferences between x and y is determined by the perpendicular bisector of the line segment from x to y – that is, all ideal points on the x side of the bisector are closer to x than y, all ideal points on the y side are closer to y than x, and all ideal points on the bisector are equidistant from x and y. If the median line perpendicular to the line through x and y is on the x side of the perpendicular bisector, it follows that more than half the ideal points are on the x side of the bisector, so x beats y. Conversely, if the median line is on the y side, y beats x. If, fortuitously, the perpendicular bisector is also the median line perpendicular to the line through x and y, x ties y, unless, even more fortuitously, two (or a larger even number of) ideal points lie on the median line (in which event x beats y or y beats x depending on how the remaining odd number of ideal points are distributed on either side of the median bisector).[5]

Now consider any point x in the space and any line L. Drop a perpendicular line of length d from x to L, intersecting L at the point x' (which is called the *projection* of x on L); if L happens to pass through x, then $x'=x$. Now project the line segment from x to x' an equal distance d beyond L to the point x^* that we call the *reflection* of x through L (x is also the reflection of x^* through L); if L happens to pass through x, then $x^*=x$. Call the *line segment* from x to x^* the *reflection line* of x (or x^*) through L.

Our interest focuses particularly on reflections through median lines. Indeed, the following is fundamental.

LEMMA 3. *Let y be any point other than x and x^* on the reflection line of x through any median line M. Then y beats x.*

Proof. The median line M is the perpendicular bisector of the reflection line of x through m. Point y lies somewhere strictly between x and x^*. Thus the perpendicular bisector of the line segment from x to y lies on the x side of M. By Lemma 2, y beats x.

5. Given majority rule in the absolute sense, x and y tie in any event. This indicates how absolute majority rule leads to slightly cleaner results.

4. The Character of Win Sets

The preceding lemmas provide the tools for characterizing the win set of an arbitrary point x in the space.

THEOREM 1. *For any line L through x*:

1. *if x is beaten by any point y on L, it is beaten by every point on L between x and y*;
2. *if x is beaten by points on L on one side of x, x is not beaten by any points on L on the other side of x; and*
3. *x is tied by at most one point on L and only if it is beaten by points on L on the same side of x.*

Proof. By Lemma 3, x is beaten by all points on a line L through x that lie between x and its reflection x^* through the median line perpendicular to L. By Lemma 2, x is beaten only by these points on L.Points (1) and (2) then follow immediately. And by Lemma 2, on any line L through x, x may be tied only by its reflection through the median line perpendicular to L. Thus, if no point on L on one side of x beats x (i.e. if the perpendicular median line does not intersect L on that side of x), no such point ties x either. This establishes (3).

Note that (1) implies that if x is beaten by any points, it is beaten by some neighboring points. Note also that the converse of (2) does not hold; if and only if the median line perpendicular to L passes through x, no points on L (on either side of x) beat x. Finally note that by (3), we can use interchangeably the phrases 'x is unbeaten' and 'x beats every point', when referring to points on a line L to one side of x.

A set $W(x)$ is *starlike* about x if and only if $W(x)$ includes all points lying on any straight line between x and any point in $W(x)$. A set $W(x)$ is *polarized* about x if and only if, when points on a line through x on one side of x belong to $W(x)$, no points on the line on the other side of x belong to $W(x)$. A set X is *thin* if it has no interior, i.e. if any neighborhood of any point in X includes points not in X. (In a two-dimensional space, any line is a thin set.)

Considering the entire space, not just points on one line L through x, we can restate Theorem 1 to characterize properties of any win set $W(x)$ in terms of the preceding definitions.[6]

THEOREM 1′. *For any point x*:

1. *W(x) is starlike about x*;
2. *W(x) is polarized about x; and*

6. Theorem 1′ holds for more general preferences (see McKelvey, 1986; and Cox, 1987).

3. *the set of points that tie x is thin.*[7]

(1), (2) and (3) in Theorem 1′ merely restate the corresponding statements in Theorem 2. Figure 2 illustrates these properties. On an arbitrary line L through x, there is just one point z that ties x, every point on L between z and x beats x, and x beats every point on L on the opposite side of x from z as well as every point on L beyond z.

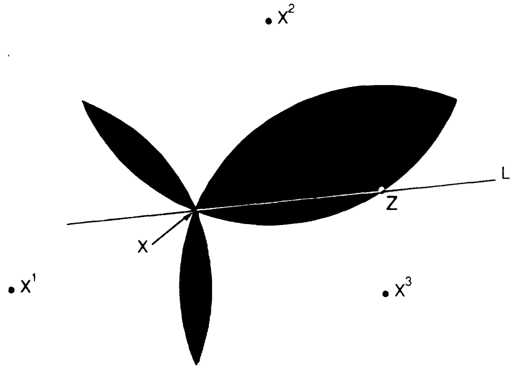

Figure 2. A Win Set

5. Conditions for an Unbeaten Point

The preceding lemmas also make it clear why an unbeaten point almost never exists in a space of two or more dimensions – that is, why $W(x)$ is almost always non-empty for all points x.

THEOREM 2. *A point x is unbeaten if and only if every median line passes through x.*

Proof. Sufficiency follows from Lemma 2. If every median line passes through x, then for any y distinct from x, the median line perpendicular to the line through x and y is closer to x than to y (for indeed it passes through x), so x beats y.

Necessity follows from Lemma 3. If x lies off *any* median line M, it has a reflection line of positive length through M, and any point on this reflection line between x and x^* beats x.

The following is an immediate corollary of Theorem 2.

7. $W(x)$ is an 'open set' – that is, a set that does not include its boundary (though there are some complexities due to the considerations discussed in footnote 11). The boundary of $W(x)$ is formed by the tie set of x. The 'closure' of $W(x)$ is the set of points that beat or tie x, i.e. the union of $W(x)$ with its boundary.

COROLLARY 2.1. *There is at most one unbeaten point.*
While with unrestricted preferences there might be several unbeaten points (that tie each other), such a situation cannot arise given the present assumptions.

It is worthwhile visualizing what a configuration of ideal points must look like if the condition specified in Theorem 2 is to hold. We can do this by deriving a series of further implications from the condition stated in the theorem.

COROLLARY 2.2. *If point x is unbeaten, every line through x is a median line.*[8]

Proof. Consider any line L through x and the line L' through x and perpendicular to L. By Lemma 1, there is (for n odd) a unique median line M perpendicular to L'. Since every median line passes through x, it must be that $M = L$, so L is a median line.

COROLLARY 2.3. *If point x is unbeaten, x is an ideal point.*

Proof. Since every line through x is a median line, there are an infinite number of median lines through x. But there are only a finite number of ideal points, so only a finite number of median lines through x can pass through ideal points other than x. Since (with n odd) every median line must pass through some ideal point, it must be that x is an ideal point.

COROLLARY 2.4. *If point x is unbeaten, x is the unique median of all ideal points that lie on each line through x.*[9]

Proof. Let k (where $1 \leqslant k \leqslant n$) be the number of ideal points on line L through x. (Usually, of course, $k = 1$, and almost always $k \leqslant 3$.) We consider two cases: (1) k is odd and (2) k is even. We show that no more than $(k-1)/2$ (if k is odd) or $k/2-1$ (if k is even) ideal points can lie on L on either side of x. (If k were even and exactly $k/2$ points lay on the same side of x, x would be a median point but not the unique median.)

(1) The number of ideal points not on L is $n-k$; since n is odd and k is odd, $n-k$ is even. The number of ideal points that lie on the side of L that has the most ideal points is at least $(n-k)/2$; let W_1 be this set of at least $(n-k)/2$ points, and let W_2 be the set of no more than $(n-k)/2$ ideal points on the other side of L. Suppose, contrary to Corollary 2.4, that at least $(k+1)/2$ ideal points lie on L on the same side of x. We now rotate L infinitesimally about x to generate a new line L' through x. The rotation can be so slight that the division of the $n-k$ ideal points into two sets W_1

8. If every line through x is a median line, x is called a *total median*. Thus another version of this theorem says that x is unbeaten if and only if x is a total median; cf. Davis et al. (1972) and Hoyer and Mayer (1974).

9. Yet another version of the theorem says that x is unbeaten if and only if x is the median of *all projected* ideal points on every line through x (Feld and Grofman, 1987). For a generalization of this version to more general preferences, see Cox (1987).

and W_2 on either side of L remains unchanged vis-a-vis L'. But even the slightest rotation means that none of the $k-1$ ideal points on L, other than x, lies on L'. We rotate in the direction so that the $(k+1)/2$ or more ideal points on L on the same side of x are placed on the same side of L' as the set W_1. Thus there are at least $[(n-k)/2]+[(k+1)/2] = (n+1)/2$ ideal points on the same side of L'. But this is impossible, since L' passes through x and is therefore a median line. Thus no more than $(k-1)/2$ ideal points can lie to one side of x on any line L through x, and x must be the unique median ideal point on L.

(2) The number of ideal points not on L is $n-k$; since n is odd and k is even, $n-k$ is odd. The number of ideal points that lie on the side of L that has the most ideal points is at least $(n-k+1)/2$. Again let W_1 designate this set of at least $(n-k+1)/2$ points and W_2 the remaining set of no more than $(n-k-1)/2$ points. Suppose, contrary to Corollary 2.4, that at least $k/2$ ideal points lie on L on the same side of x. As before, we rotate L infinitesimally about x to generate a new line L' through x. And we rotate in the direction so that the at least $k/2$ ideal points on L on one side of x are placed with the set W_1. Thus there are at least $[(n-k+1)/2]+k/2 = (n+1)/2$ ideal points on the same side of L', which again leads to a contradiction. Thus no more than $k/2-1$ ideal points can lie to one side of x on any line L through x, and x must be the unique median ideal point on L.

Thus, if the condition specified in Theorem 2 is to hold, and if all voter preferences are diverse – in particular if the unbeaten point is the ideal point of only one voter, there must be one ideal point x_i such that all remaining ideal points can be paired off in such a way that the two points in each pair lie on a straight line with, and on opposite sides of, x_i. (Two or more such pairs may lie on the same straight line.) Figure 3 illustrates such a configuration of ideal points for $n=7$. For the case of Euclidean preferences, this constitutes the Plott (1967) *pairwise symmetry condition* sufficient for the existence of an unbeaten point. For n odd and diverse preferences, this symmetry condition is also necessary for an unbeaten point. But if preferences are not diverse – in particular, if the unbeaten point is the shared ideal point of two or more voters – (or if n is even), such symmetry is not necessary.[10] Figure 4 provides an illustration, where the shared ideal point of voters 4 and 5 is unbeaten.

6. Win Sets with an Unbeaten Point

Given an arbitrary point x, we can always demarcate the win set $W(x)$ by examining every line L through x, provided that we can determine where

10. That the Plott symmetry condition is not more generally necessary for the existence of an unbeaten point is quite often misunderstood, as Enelow and Hinich (1983) observe.

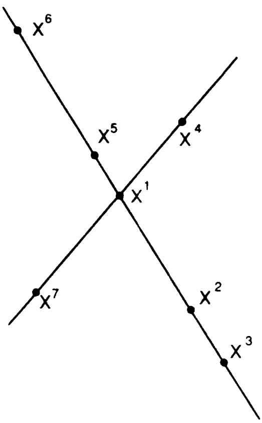

Figure 3. The Plott Symmetry Condition

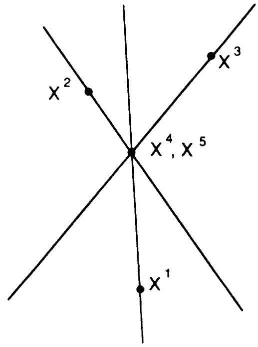

Figure 4. An Unbeaten Point with Non-Diverse Preferences

the median line M perpendicular to L intersects L. By Lemma 3, point x is beaten by every point y on L between x and its reflection x^* through M; by Lemma 2, x is beaten by only these points (and possibly by x^*) on L.

Proceeding in this manner, we can establish the following theorem in the special case in which an unbeaten point exists; it is essentially a theorem due to Davis, DeGroot and Hinich (1972).[11]

THEOREM 3. *Given an unbeaten point c, if point y is further from c than point x is, x beats y.*

Proof. Since there is an unbeaten point c, from Theorem 2 every median line passes through c. Consider any point x at a distance d from c and any line L through x, as shown in Figure 5. M is the median line perpendicular to L. By Lemma 3, x is beaten by every point on L between x and its reflection x^* through M and, by Lemma 2, x is beaten only by these points (and possibly by x^* itself) on L. Since M is the perpendicular bisector of the reflection line, x and x^* are equidistant from all points on M, including c. Thus the distance from x^* to c is d, and the distance from c to any point on L between x and x^* is less than d (as is shown clearly in Figure 5 by drawing in part of the circle with center c and radius d). So, in any event, x is beaten by every point on L that is closer to c than

11. The Davis, DeGroot and Hinich theorem says that, if there is an unbeaten point, majority preference is transitive. (This certainly is not true in the non-spatial case.) Theorem 3 would be equivalent if it said that 'y beats x if *and only if* y is closer to c'. This in turn would imply that points equidistant from c must tie. We can assure this by making either of two changes in our assumptions. The first is to define majority rule in the absolute sense; cf. footnote 5 (this is what Davis, DeGroot and Hinich do). The second is to specify that preferences are *diverse* and, in particular, that only one voter has c as his ideal point.

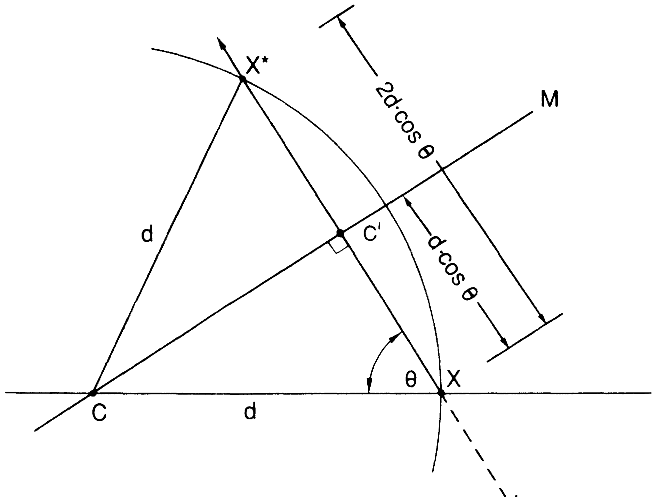

Figure 5. Demarcating a Win Set with an Unbeaten Point

x is (and beats every point on L that is further from c than x is). Restating these conclusions for all lines through x gives the theorem.

Thus, if there is an unbeaten point c, any win set $W(x)$ is the set of points enclosed by the circle centered on c and passing through x.[12]

We can also derive a simple formula for the distance from x to the boundary of its win set in any direction, i.e. from x to x^* on any line through x. A *ray from a point*, x, is a half line beginning at x and pointing outward from x in any direction. We may specify a ray from point x in terms of the angle Θ between the given ray and the ray from x through c (see Figure 5; in fact there are two rays for each Θ: the one drawn in the figure and the one in the mirror image of the figure below the line through c and x).[13] Note that x, c and the projection c' of c on L form the vertices of a right triangle. Recall that the *cosine* of an angle in a right triangle is the ratio of the length of the adjacent side to the length of the hypotenuse. Thus if p is the distance from x to c', $\cos \Theta = p/d$, so $p = d \cdot \cos \Theta$, and the distance from x to x^* (i.e. the distance from x along the ray specified by Θ to the boundary of its win set) is $2p = 2d \cdot \cos \Theta$. The locus of points at a distance $2d \cdot \cos \Theta$ from x is the circle centered on c and passing through x.

12. $W(x)$ may possibly include some points on the circle – again unless we assume absolute majority rule or require preferences to be diverse.

13. This is a strictly two-dimensional statement. In three (or more) dimensions there are an infinite number of rays from x for a given Θ (defining a cone with vertex at x), but our conclusion generalizes in the natural way, i.e. $W(x)$ is enclosed by a sphere centered on c and passing through x.

7. The Yolk

Theorem 2 and the subsequent discussion should persuasively indicate that, in two (or more) dimensions, there is almost never an unbeaten point, for such a majority rule equilibrium requires that ideal points 'line up' in a highly unlikely fashion. Put more directly in terms of Theorem 2, it is simply unlikely that ideal points will be distributed in such a way that all median lines intersect exactly at a common point. But, at the same time, it does seem that ideal points will typically be distributed in such a fashion that each median line would cut more or less through the center of the distribution, so that there would be a fairly small region (though not a single point) through which all median lines pass. It would be useful to have some measure of the size of this region, which would then indicate how far the distribution of ideal points departs from one that would generate an unbeaten point.

Following Ferejohn et al. (1984) and McKelvey (1986), we define the *yolk* as the region bounded by the circle of minimum radius that intersects every median line. The yolk for the configuration of ideal points displayed in Figure 1 is shown in that diagram. Notice that a circle that intersects every limiting median line necessarily intersects every other median line as well, as non-limiting median lines through any point x lie between pairs of limiting median lines through x. McKelvey (1986) provides a linear programing method for computing the exact location and size of the yolk, i.e. its *center*, c, and *radius*, r; but the location and size of the yolk can be determined with fair accuracy on the basis of visual inspection once limiting median lines are drawn in.

The location of the yolk may be thought of as indicating the generalized center (in the sense of the median) of the distribution of ideal points. It is possible for the yolk to be a circle with zero radius, i.e. the single point c, as shown in Figure 6(a). Obviously, this is true if and only if all median lines pass through a common point. Theorem 2 tells us that in this case (and only in this case) the common point is unbeaten, and the discussion in the previous section applies. In general, the yolk is enclosed by the smallest circle such that every median line passes through it. Thus the radius of the yolk may be thought of as measuring the extent to which the configuration of ideal points departs from one that generates an unbeaten point. In Figure 6(b), the configuration displayed in Figure 6(a) has been slightly perturbed, creating a yolk of positive but small radius. In Figure 6(c), the configuration has been further perturbed (in a deliberately contrived fashion), creating a relatively large yolk.

It is important to emphasize that the size of the yolk does *not* indicate, and is *not* a function of, the dispersion of voter ideal points, except in a limited way. It is true that the yolk must be contained within the distribution of ideal points. Thus if all ideal points are concentrated in a small area, the yolk also must be small. On the other hand, if the ideal points

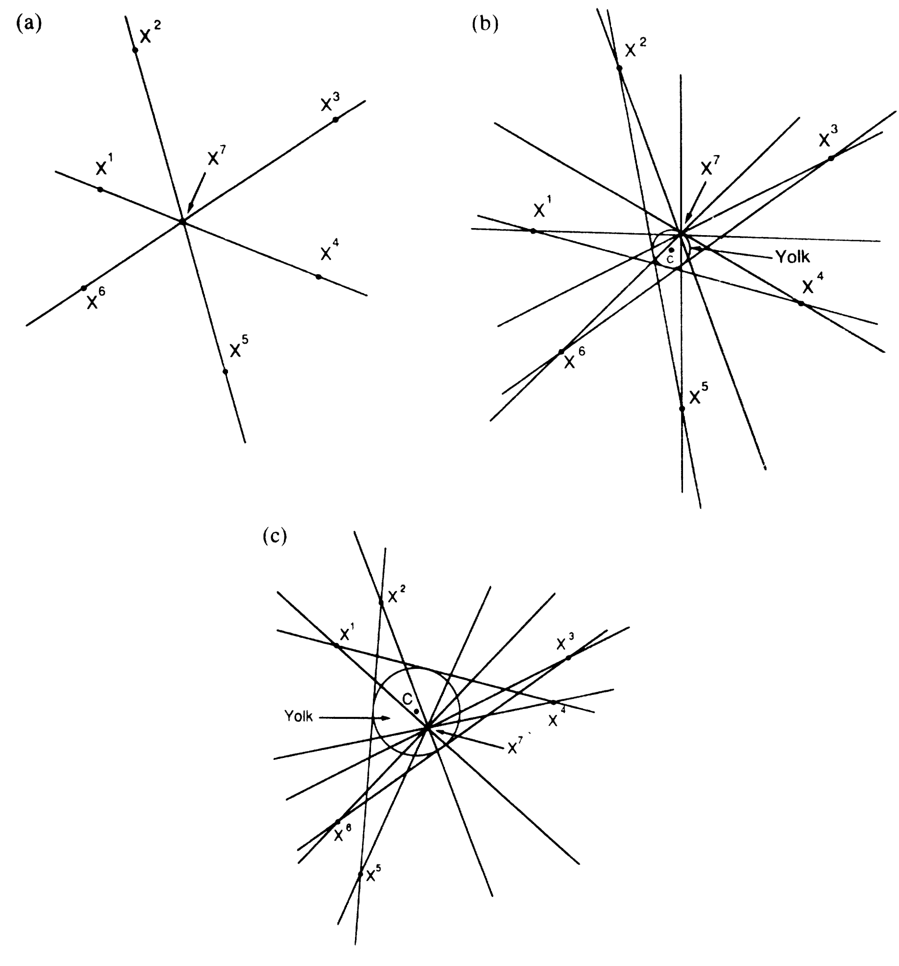

Figure 6. (a) Yolk with Zero Radius; (b) Yolk with Small Radius; (c) Yolk with Large Radius

are spread out, the yolk may be either small or large, depending on the particular configuration of dispersed ideal points. This is illustrated in Figures 6(a)–(c), in which the configurations of seven ideal points all exhibit essentially the same dispersion, but the yolks vary greatly in size. (Note that these figures display limiting median lines only.)

We now present a useful lemma concerning the yolk.[14]

14. This is a strictly two-dimensional proposition. In three dimensions median 'lines' become median planes, the yolk is a sphere, and at least four median planes are tangent to the yolk. In w dimensions, at least $w - 1$ median hyperplanes are tangent to the yolk. In any event, Lemma 5 generalizes straightforwardly.

LEMMA 4. *In the absence of an unbeaten point, at least three median lines are tangent to the yolk.*

Proof. It is a basic result of geometry that a circle can be inscribed within any triangle, so that each of the three sides of the triangle is tangent to the circle. Any three limiting median lines (that do not intersect at a common point) enclose a triangle, and the circle inscribed within this triangle is clearly the smallest circle intersecting all three median lines. Consider a fourth limiting median line. If it does not intersect the circle, the additional median line together with two of the original three form a triangle whose inscribed circle intersects all four median lines, and which is the smallest circle to do so. So, in any event, the smallest circle intersecting all four median lines is tangent to at least three of them. And so forth, until we have considered every limiting median line (which are finite in number). As previously noted, this circle must intersect all non-limiting median lines as well.

Lemma 4 has this further implication.

LEMMA 5. *In the absence of an unbeaten point, for any point x, there is some median line M such that x and the center of the yolk c lie on the same side of M.*

Proof. Refer to Figure 7, which shows the yolk with center c and the three median lines known to be tangent to the yolk. However these median lines are drawn, they partition the space into three subsets labelled A, B and C. (For precision, we should specify that B includes the lines themselves.) It can be checked that, whether x belongs to A, B or C, it is true that x and c lie on the same side of at least one of these three median lines.

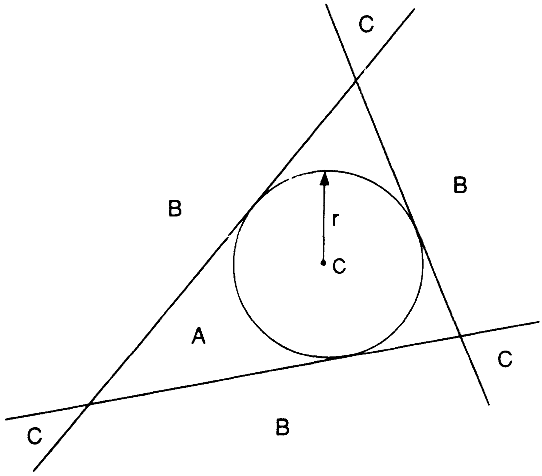

Figure 7. Construction Used in Lemma 5

8. Win Sets without an Unbeaten Point

The following theorem complements Theorem 3, and it pertains to the generic case – that is, the case that almost always obtains – in which there is no unbeaten point and the yolk has positive radius.

THEOREM 4. *In the absence of an unbeaten point, for any point x there is some other point y that both beats x and is further from the center of the yolk than x is.*

Proof. Lemma 5 tells us that, for any point x, there is some line L passing through x such that the median line M perpendicular to L lies beyond the center of the yolk, as shown in Figure 8, where e is the perpendicular distance from c to M.[15] As we saw in connection with Theorem 3 (and Figure 5), if x could not be beaten by points more distant than x is from the center of the yolk, the point on L most distant from x that could beat x would be x'', the reflection of x through the line passing through c and perpendicular to L (at a distance of $2d \cdot \cos \Theta$ from x). But in fact x is beaten by every point on L up to (and possibly including) x^*, the reflection of x through the median line perpendicular to L (at a distance of $2d \cdot \cos \Theta + 2e$ from x). Thus x^*, further from c than x is, demarcates the boundary of $W(x)$ along L.

Reversing the roles of the two points, it follows also that for any point x there is some other point z that x beats and that is closer to the center of the yolk than x is.

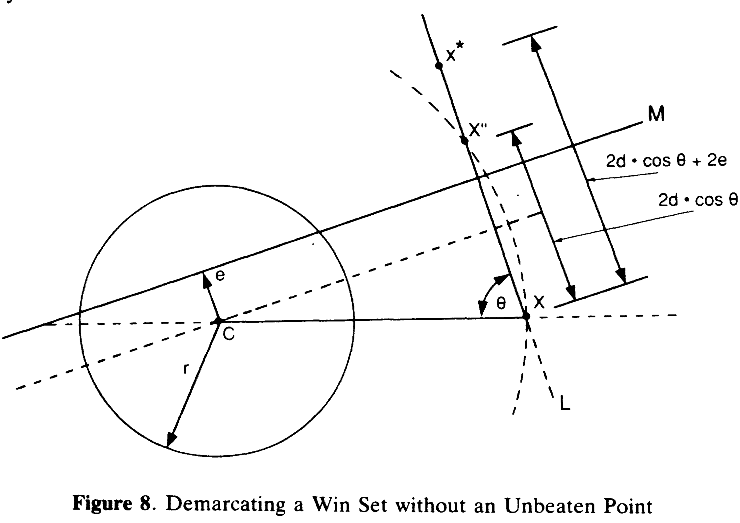

Figure 8. Demarcating a Win Set without an Unbeaten Point

15. Actually, Lemma 4 gives us the stronger result that we can find some line L through x such that the median line perpendicular to L lies beyond c and is tangent to the yolk. In this case, $e = r$.

Thus, in the absence of an unbeaten point, the win set $W(x)$ of a point x is never contained in the circle centered on the center of the yolk and passing through x (as is always the case given an unbeaten point). $W(x)$ always extends beyond this circle in some places (and falls short of it in other places).

This result essentially drives McKelvey's (1976) 'global cycling' theorem. By repeated application, it says that we can construct a majority preference trajectory of this form: x is beaten by y, y is beaten by z, z is beaten by v, and so forth, such that each new point in the trajectory is further from the center of the yolk than the preceding point. In this way, the trajectory can move outward from the center of the yolk without limit. It is easy to believe that, if the trajectory moves far enough outward, it can always move back in to x to complete the cycle. We are able formally to prove this as a corollary to a subsequent theorem.

9. Win Sets and Distance from the Yolk

As noted at the beginning of Section 6, given an arbitrary point x at a distance d from the center of the yolk c, we can demarcate $W(x)$ by examining every line L through x and determining where the median line M perpendicular to L intersects L.[16] We did just this in the case in which c is unbeaten, i.e. c is the only point in the yolk.

Of course, if the only information we have concerning the configuration of ideal points is the size and location of the yolk (i.e. that information conveyed by the parameters c and r), we do not know exactly where the median line M perpendicular to L lies. But we do know that it lies between the two *tangent lines*, T_1 and T_2, perpendicular to L and tangent to opposite sides of the yolk (see Figure 9), for by definition every median line passes through the yolk.

If T_1 and T_2 both intersect L on the same side of x (which can be true only if x lies outside of the yolk), as in Figure 9(a), M *must* lie on that side of x, so (regardless of the particular configuration of ideal points) x *must* be beaten by points on L on that side of x. And by the polarity property of win sets (Theorem 1′), x *cannot* be beaten by any points on L on the other side of x.

If T_1 and T_2 intersect L on opposite sides of x (which must be true if x lies inside the yolk), as in Figure 9(b) we cannot say on which side of x

16. We might also observe that, if we know the shape of the win set $W(x)$ of *any point x*, we can deduce the location of all median lines. (Each median line is perpendicular to some ray from x and bisects the line segment from x to the boundary of $W(x)$.) In turn, we can then deduce the shape of the win set $W(y)$ of *any other* point y. *A single win set, therefore, contains complete information concerning majority preference over the entire alternative space* (given that individual preferences are Euclidean). We cannot recover the location of all individual ideal points from a win set, however.

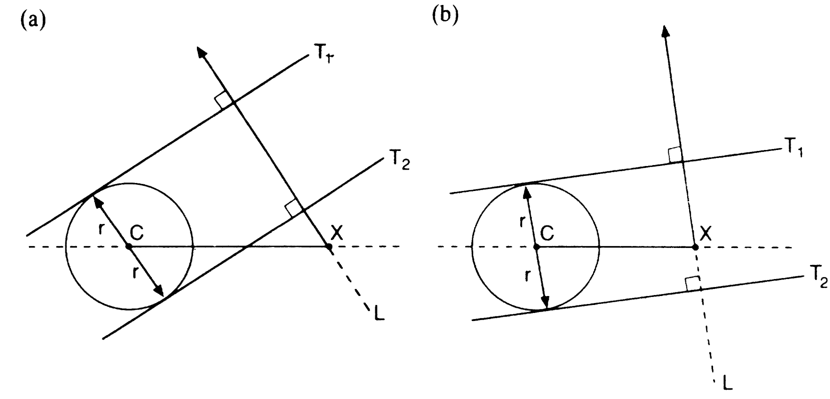

Figure 9. (a) Both Tangent Lines Intersecting a (Dominating) Ray; (b) Only One Tangent Line Intersecting a (Contingent) Ray

M lies, but (unless it happens that M passes through x, in which event no point on L beats x) x is beaten by points on L on *one or other* side of x (depending on the particular configuration of ideal points), but in any case (by polarity) not both sides.

Thus, given the parameters c (the center of the yolk), r (the radius of the yolk) and d (the distance from c to x), we can partition all *lines* through x into two classes, according to whether T_1 and T_2 intersect the line on the same side of x or not.

In turn, we can partition all *rays* from x into three classes:

1. *dominating* rays, which must intersect $W(x)$ regardless of the particular configuration of ideal points, because both T_1 and T_2 strictly intersect each such ray;[17]
2. *dominated* rays, which cannot intersect $W(x)$ regardless of the particular configuration of ideal points, because neither T_1 nor T_2 strictly intersect each such ray; and
3. *contingent* rays, which may or may not intersect $W(x)$ depending on the particular configuration of ideal points, because either T_1 or T_2, but not both, strictly intersects each such ray.

We find it useful to call a ray *undominated* if it is either dominating or contingent; we adopt the convention (observed in Figures 9(a) and 9(b)) that, in the event that a ray is dominating, T_2 lies closer to x than T_1 and, in the event that a ray is contingent, the ray intersects T_1 but not T_2.

We call two rays *opposites* if they lie on the same line pointing in opposite directions. Then, from the polarity property, if a ray from x is dominating, its opposite is dominated, and vice versa; and if a ray is

17. By 'strictly intersecting' a ray from x, we mean that T passes through a point on the ray other than x itself.

contingent, so is its opposite. [18] Thus the sets of dominating and dominated rays constitute the two nappes of a cone with the vertex at x and centered on the line through c and x. These sets constitute the *dominating cone* and the *dominated cone*, respectively, with respect to x; the dominating cone faces toward the yolk and the dominated cone faces away from the yolk. The set of contingent rays is the complement of these cones; we may call it the *chaotic region*. These definitions are illustrated in Figure 10.

The next question is how to specify which rays are of which type, and thus to specify exactly where these regions about x lie. (We have already observed that, if point x is inside the yolk, all rays from x are contingent; thus the chaotic region with respect to such a point fills the entire space.) As before, we specify rays from x in terms of the angle Θ between the ray in question and the ray from x through the center of the yolk. We can answer the question of which rays are of which type by computing the critical angles Θ^* and Θ^{**} that separate dominating from contingent rays and contingent from dominated rays, respectively, as shown in Figure 10. Each angle is a function of r, the radius of the yolk, and d, the distance from x to c. Angle Θ^* defines the ray such that T_2 passes through x (see Figure 11(a)); angle Θ^{**} defines the ray such that T_1 passes through x, and is simply $180° - \Theta^*$ (see Figure 11(b)). Thus $\cos \Theta^* = r/d$ and $\cos \Theta^{**} = -r/d$.

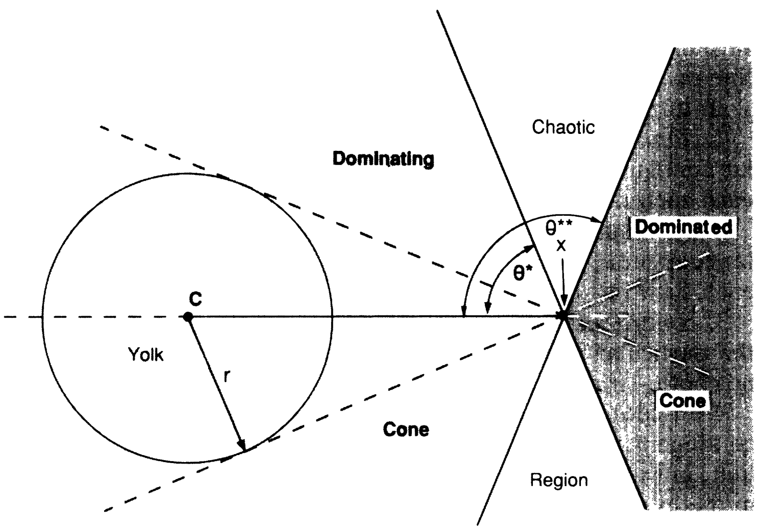

Figure 10. Dominating, Chaotic, and Dominated Regions About a Point Outside the Yolk

18. One 'boundary condition' constitutes an exception to these statements: if a tangent line passes exactly through x, one ray from x is contingent and its opposite is dominated.

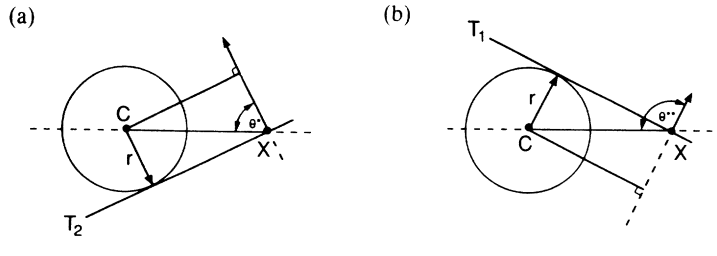

Figure 11. (a) Critical Angle Separating Dominating and Chaotic Regions; (b) Critical Angles Separating Chaotic and Dominated Regions

We can summarize this discussion in the following theorem. (Recall that cos Θ decreases as Θ increases and that, in particular, $\cos 0° = 1$, $\cos 90° = 0$, and $\cos 180° = -1$.)

THEOREM 5. *For any point x at a distance of d from the center of the yolk c, and for any ray from x specified by Θ:* (1) *if* $1 \geqslant \cos \Theta > r/d$, *the ray is dominating;* (2) *if* $r/d < 1$ *(i.e. if x is outside the yolk) and* $-r/d \geqslant \cos \Theta \geqslant -1$, *the ray is dominated; and* (3) *if* $r/d \geqslant \cos \Theta > -r/d$ *or if* $r/d \geqslant 1$ *(i.e. if x is in the yolk), the ray is contingent.*

Proof. Follows from the preceding discussion.

Given these relationships, we can readily see what happens to the three regions of the space about *x* as the ratio *r*/*d* changes.

As *x* moves further from the yolk (as *d* increases), or the radius of the yolk shrinks (as *r* decreases), so that the ratio *r*/*d* decreases and approaches zero, the critical angles Θ* and Θ** approach 90° (from below and above, respectively). Thus the dominating and dominated cones widen and the chaotic region contracts. Therefore, as distance from the yolk increases or the size of the yolk decreases, majority rule becomes more orderly – in that a given point is beaten by a larger and larger fraction (approaching 100 per cent) of nearby points in the direction of the yolk and by a smaller and smaller fraction (approaching 0 per cent) of points in the direction away from the yolk. (As we have seen from Theorem 3, majority rule becomes perfectly orderly in this sense when the radius of the yolk shrinks to zero.)

As *x* moves closer to the yolk (as *d* decreases), or the yolk expands (as *r* increases), so that the ratio *r*/*d* increases and approaches one, the critical angles Θ* and Θ** approach 0° and 180° respectively, so the dominating and dominated cones narrow and the chaotic region expands. Therefore, as distance from the yolk decreases or as the size of the yolk increases, majority rule becomes more chaotic – in that a given point is beaten by a smaller and smaller fraction (approaching 50 per cent) of nearby points in the direction of the yolk and by a larger and larger fraction (approaching 50 per cent) of points in the direction away from the yolk.

As we have already seen, if x is within the yolk, all rays from x are contingent and, in this sense, majority rule within the yolk is totally chaotic – a point within the yolk may be beaten by nearby points 'on all sides'.

Next, we should bear in mind that, while x beats *every* point on a dominated ray, x certainly is *not* beaten by every point on a dominating ray but only by 'nearby' points. Put otherwise, while $W(x)$ includes no points in the dominated cone, $W(x)$ includes only all 'nearby' points in the dominating cone and some 'nearby' points in the chaotic region. The question naturally arises of how 'nearby' these points must be. The answer to this question follows directly from previous considerations.

Consider any dominating ray from x (see Figure 12). By definition both T_1 and T_2 strictly intersect the ray. The median line perpendicular to the ray cannot be closer to x than T_2. Thus x must be beaten by all points on the ray between x and its reflection through T_2. The distance from x to c' (the projection of c onto the ray) is, as we saw earlier, $d \cdot \cos \Theta$. The projection x' of x onto T_2 is closer to x by the distance r, so the distance from x to x' is $d \cdot \cos \Theta - r$. The distance from x to its reflection x_2^* through T_2 is just twice this, i.e. $2d \cdot \cos \Theta - 2r$. (Notice that this

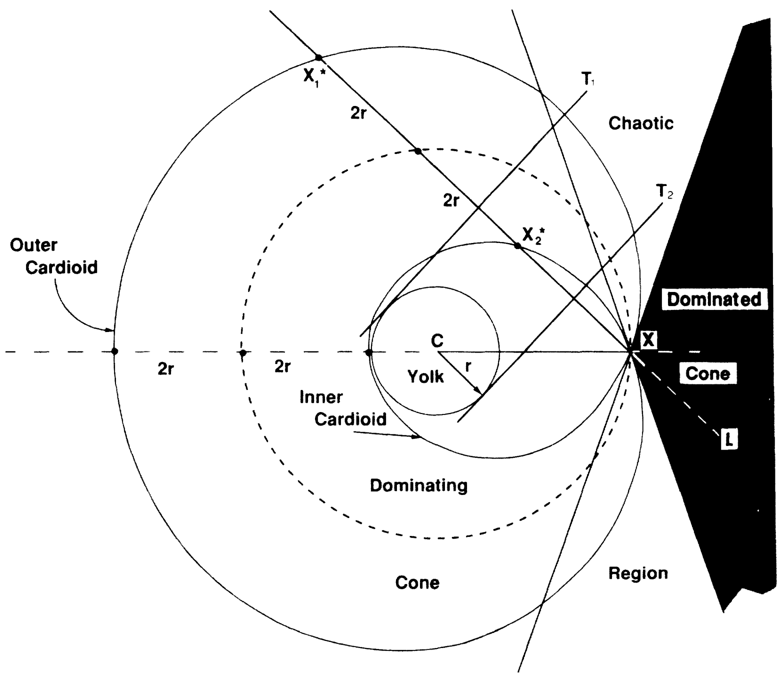

Figure 12. Cardioid Bounds on a Win Set

expression is positive just so long as the ray is dominating, i.e. cos $\Theta > r/d$.) Thus x must be beaten by all points on a dominating ray up to a distance of $2d \cdot \cos \Theta - 2r$ from x.

Now consider any undominated ray from x. By definition, T_1 strictly intersects the ray. The median line perpendicular to the ray cannot be further from x than T_1. Thus x must beat all points on the ray beyond x and its reflection through T_1. Calculating in the same manner as above, the distance from x to its reflection x_1^* through T_1 is $2d \cdot \cos \Theta + 2r$. (Notice that this expression is positive just so long as the ray is undominated, i.e. $\cos \Theta > -r/d$.) Thus x must beat points on an undominated ray beyond a distance of $2d \cdot \cos \Theta + 2r$ from x.

In summary, we have established the following theorem.

THEOREM 6. *For any point x at a distance of d from the center of the yolk c, and for any ray from x specified by* Θ: (1) *x is beaten by all points on a dominating ray up to a distance of* $2d \cdot \cos \Theta - 2r$ *from x;* and (2) *x beats all points on an undominated ray beyond a distance of* $2d \cdot \cos \Theta + 2r$ *from x.*

Proof. Follows from the preceding discussion.

This is, in effect, the theorem due to Ferejohn et al. (1984) who state it in the following manner. Recall that the locus of points at a distance of $2d \cdot \cos \Theta$ from x is simply the circle centered on c and passing through x. From the analysis just above, inner and outer bounds on $W(x)$ are given by the locus of points at a distance of $2d \cdot \cos \Theta - 2r$ and at a distance of $2d \cdot \cos \Theta + 2r$, respectively, from x. The first locus is the *cardioid* with center c, underlying radius d, its *cusp* at x, and (negative) *eccentricity* of $-2r$; the second locus is an otherwise similar cardioid with (positive) eccentricity of $+2r$. Such cardioids are shown in Figure 12. Ferejohn et al. state the theorem this way: the region enclosed by the inner cardioid (with negative eccentricity) is contained in $W(x)$, and $W(x)$ in turn is contained in the outer cardioid (with positive eccentricity).[19]

10. Cycle Lengths and the Uncovered Set

Given the preceding results, we can now readily derive McKelvey's 'global cycling' theorem. We are also in a position to determine how long a cycle must be to include two arbitrary points, a consideration that has direct relevance also for the size of the uncovered set (since, if a point z beats an uncovered point x, there is some third point y such that x beats y and

19. If x is inside the yolk (so no rays are dominating), the inner cardioid disappears and the outer cardioid does not intersect x. If x is at c, the outer cardioid is simply the circle centered on c with radius $2r$. We can use Lemma 4 to show that the boundary of $W(x)$ must touch the outer cardioid at some point (and likewise the inner cardioid, if it exists).

y beats z). Thus we can also derive McKelvey's circular bound on the uncovered set.

Consider a point x at distance d from the center of the yolk c. Let us construct two circles, both centered on c and with radii of $d-2r$ and $d+2r$ respectively. (The first circle will exist only if $d > 2r$.) It is apparent from Figure 13, and from the discussion in the preceding section, that the region enclosed by the smaller circle is a subset of the region enclosed by the inner cardioid that is an inner bound on $W(x)$, and the region enclosed by the larger circle is a superset of the region enclosed by the outer cardioid that is an outer bound on $W(x)$. Thus these are circular (inner and outer) bounds on $W(x)$; x beats every point in the smaller circle and is beaten by every point outside the larger circle.

It is worth stating this formally, as a corollary to Theorem 6.

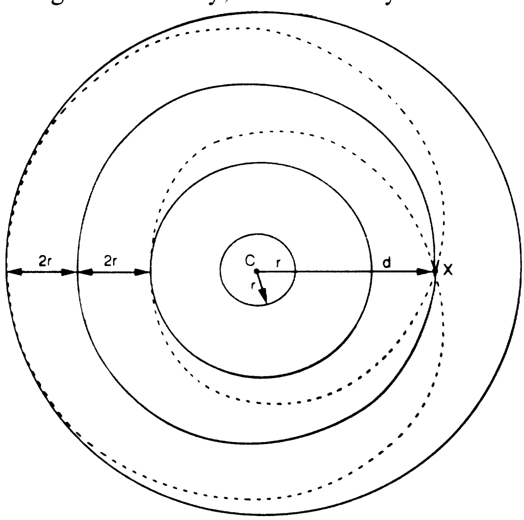

Figure 13. Circular Bounds on a Win Set

COROLLARY 6.1. *If point y is more than 2r further from the center of the yolk than point x is, x beats y.*

Note that this corollary subsumes Theorem 3, which pertains to the special case in which $2r = 0$.

Given this corollary in conjunction with Theorem 4, we can prove McKelvey's (1976) 'global cycling' theorem.

THEOREM 7. *In the absence of an unbeaten point, for any pair of points x and y, we can find a majority preference cycle including both x and y.*

Proof. Suppose that x is beaten by y. By repeated application of Theorem 4, we can always construct a trajectory of this form: y is beaten by z, z is beaten by w, and so forth, to a point at any finite distance from the center of the yolk. In particular, we can construct such a trajectory to some point v at a distance greater than $d+2r$ from the center of the yolk, where d is the distance from x to the center of the yolk. By Corollary 6.1, v is beaten by x. Thus we have the required cycle including both x and y. If y is beaten by x, we can construct a similar trajectory from x to y to get the required cycle. If x and y tie, we can construct similar trajectories from x to y and from y to x and put them together to get the required cycle. This establishes McKelvey's theorem.[20]

While Corollary 6.1, in conjunction with Theorem 4, establishes McKelvey's theorem, it also indicates a majority preference trajectory leading from point x to a point y; considerably more distant from the center of the yolk may require many intermediate steps, especially if the yolk itself is small, since each step in the trajectory can lead *at most* $2r$ further out from the center of the yolk.[21] We can state this formally.

COROLLARY 6.2. *Given any two points x and y linked in a majority preference cycle including k points altogether, where d_1 and d_2 are the distances of x and y, respectively, from the center of the yolk and where d_1 is less than d_2, d_2-d_1 cannot exceed $2r(k-1)$.*

We also have the following corollary.

COROLLARY 6.3. *If point y is more than $4r$ away from the center of the yolk than point x is, x covers y.*

Proof. Let d_1 and d_2 be the distances of x and y, respectively, from the center of the yolk. Since d_2-d_1 is greater than $4r$, by Corollary 6.1 x beats y, and the circle centered on c with radius d_2-2r encloses the circle centered on c with radius d_1+2r. Any point z that y beats must be on or outside the larger circle (with radius d_2-2r) but then z is outside the smaller circle as well, which means x beats z. So x beats everything y beats. Any point z that beats x must be on or inside the smaller circle (with radius d_1+2r) but then z is inside the larger circle as well, which

20. However, McKelvey defines majority preference in the absolute sense. Strictly, then, we must also show that every majority preference relationship in the trajectories we have constructed can be effected by a majority of all voters (not just a majority of non-indifferent voters). All this requires is that the trajectory never move from a point w to a point u such that u is the reflection w^* of w through the median line perpendicular to the line connecting w and u but instead to a point u such that u is between w and w^*. This might possibly mean the required trajectory would be slightly longer than otherwise, but it could be constructed in any event.

21. Also, if this maximum outward movement of $2r$ is to be approached at each step, the trajectory will move wildly back and forth across the alternative space, since – as the shape of the outer cardioid indicates (see Figure 12) – the points most distant from the center of the yolk that beat a given point x tend to be located on the far side of the yolk from x. Both of these considerations suggest significant limitations on the kind of 'agenda control' that, as McKelvey noted, are implied by his 'global cycling' theorem. We have pursued this line of argument systematically in another paper (Feld et al., 1989).

means z beats y. So z does not tie anything that beats x either. By definition, then, x covers y.[22]

In Figure 13, for example, every point inside the inner circle covers every point outside the outer circle.

We finally state the theorem giving McKelvey's (1986) bounds on the uncovered set.

THEOREM 8. *The uncovered set of points is bounded by the circle centered on the center of the yolk with radius 4r.*

Proof. By Corollary 6.3, the point c at the center of the yolk covers all points outside the circle with center c and radius $4r$. Thus the set of points not covered by c is within this circular bound. The uncovered set, i.e. the set of all points none of which is covered by *any* other point, is a subset of the set of points not covered by c. So, in any event, the uncovered set lies within the same circular bound.

Thus, any political choice process that produces outcomes in the uncovered set (such as those identified by Miller, 1980; and McKelvey, 1986), produce outcomes that are generally centrally located and, if the yolk is relatively small, confined to a relatively small portion of the issue space.[23]

11. Discussion

The purpose of this paper has been essentially methodological: to present some basic results concerning majority rule on an alternative space in such a way that the meaning and proofs of the theorems are accessible to a relatively broad political science audience. But there is an underlying substantive theme to this exposition as well, and it is worth highlighting it explicitly, since it qualifies some interpretations of the 'chaos theorems'.

The first substantive point to make is that majority rule on an alternative space of two (or more) dimensions typically is not all that 'chaotic'. The strength of points with respect to majority preference is very much a function of their centrality in the space, i.e. their closeness to the center of the yolk. If there is an unbeaten point, majority rule is perfectly behaved, as the strength of points is *exclusively* a function of their centrality (Theorem 3). Otherwise (i.e. in the generic case in which there is no

22. Another way to say this is that, if y is more than $4r$ further away from c than x is, the inner cardioid with cusp at y always encloses the outer cardioid with cusp at x.

23. This raises the question as to the 'typical' size of the yolk. We expect the yolk typically to be small relative to the distribution of voter ideal points. Certainly the yolk is contained within this distribution; it can contain more than a very small proportion of the ideal points only if the remaining ideal points are very oddly distributed; and the yolk is unlikely to expand in size, and very probably shrinks in size, as new ideal points are added to a distribution. We pursue these matters in more detail in Feld et al. (1988).

unbeaten point) an element of 'imperfection' is introduced into majority rule, in that somewhat less central points sometimes beat somewhat more central points. But this is the exceptional case and in any event can occur only if the less central point is not too much further from the center of the yolk than the more central one (e.g. Corollary 6.1).

The second point to make is that the generic 'imperfection' of majority rule on a space of alternatives is itself a matter of degree, though the standard verbal statement of McKelvey's 'global cycling theorem' (used in Section 1) – that if majority rule fails at all, it fails completely – may suggest that majority rule is either perfectly behaved or totally chaotic. The degree of 'imperfection' is a direct function of the size of the yolk, which determines the size of the circular bounds identified by Corollary 6.1, determines the eccentricity of the inner and outer cardioids, and (together with the distance from point x to the center of the yolk) determines the magnitude of the critical angles Θ^* and Θ^{**} and thus the size of the chaotic region about x. If the yolk is large, majority rule is indeed quite 'imperfect' but, as the yolk decreases in size, the behavior of majority rule approaches perfection in a continuous manner.

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