Coalitions and Collective Action

Coalitions and Power in Political Situations[1])

G. Owen, and *B. Grofman*, Irvine

Abstract: *In a voting situation, we assume that voters have given positions in a "political space;" they also have given "inertia factors" which measure their resistance to a change in position. Based on these parameters, probabilities of the possible coalitions and a modified power index are defined. An example is worked out in detail.*

1 Introduction

In the past several years there has been a resurgence of interest in the concept of power in the context of the theory of committees and elections. New power indices have been proposed [*Coleman*; *Kushner/Urken*; *Deegan/Packel*; *Hofshi/Korsh*; *Joglekar*; *Packel/Deegan*] and the axiomatic and calculational foundations of familiar measures of power such as the Shapley-Shubik value [*Shapley/Shubik*] and the Banzhaf index [*Banzhaf*] have been further developed and their properties further explored [*Owen*, 1972; *Dubey*, 1975, 1976; *Brams/Affuso*; *Lucas*; *Straffin*, 1977; *Brams/Lake*; *Holler/Packel*]. In addition there has been a good deal of empirical work done which has made use of game theoretic derived notions of power. [See e.g., *Miller*; *Brams*, 1975; *Brenner*; *Walther*; *Lucas*; *Grofman/Scarrow*, 1979, 1980; *Laakso/Taagepera*, 1979; *Taagepera/Laakso*; *Holler/Kellerman*; *Frank/Shapley*]

One suggestion which has been made by several authors [e.g., *Owen*, 1971; *Deegan/Packel*; *Packel/Deegan*; *Straffin*, 1977, 1978; *Nevison*; *Grofman*, 1979; *Dubey/Shapley*] is to develop power indices which are based on assumptions as to coalitional probabilities which are more "realistic" than equiprobable combinations (the Banzhaf index) or equiprobable permutations (the Shapley-Shubik value).

In this paper we introduce a new model which improves upon that of previous research in that its coalitional dynamics reflect *both* the ideological proximity of actors in some m-dimensional Euclidean Space, and also a proxy variable for the ease with which actors can agree to join in protocoalitions – a variable which can, in general, be expected to vary with the size of the party's seat strength.

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2. The Mathematical Model

We consider here a voting-type situation in which each member, i, of a body $N = \{1,2,\ldots,n\}$ has p_i votes and is located at a position, x_i, of some m-dimensional Euclidean space R^m. We assume any coalition with at least q votes is a winning coalition.

Let a motion be presented. We can identify this motion with a position $\sigma \in R^m$. (In the simplest case, $m = 1$, and $0 \leq \sigma \leq 1$.) The distance $\|x_i - \sigma\|$ (using some type of norm on R^m) is a measure of the "resistance" of player i to the given motion.

Apart from this, we assume a given "inertia" coefficient, $\lambda_i > 0$, to player i. This is a measure of i's resistance, in general, to any sort of change.

Let α be a given constant, $0 < \alpha < \infty$. (The limiting cases, $\alpha = 0$ and $\alpha = \infty$, are also of interest.) For $0 \leq t \leq 1$, let us set

$$y_i = t^{\lambda_i \|x_i - \sigma\|^\alpha} \tag{1}$$

As t increases from 0 to 1, y_i will also increase monotonically from 0 to 1.

We interpret this model as follows: If a proposal with position σ is made, the various voters will, over a certain period of time, decide whether to approve the proposal. The probability that voter (or party) i approve the proposal by-time t is precisely $y_i(t)$. We might also assume that a limit time, T, is given, possibly at random. The proposal will pass if a winning coalition favors the proposal by time T, $0 < T < 1$. An interpretation of the parameters α and λ is given in Section III.

With this model in mind, we might ask the following questions:

(a) What coalitions will normally form?
(b) How does this model affect the several voters' power?
(c) What is the probability that a given proposal will pass?

(a) Concerning the first question, we should perhaps distinguish two cases:

(i) The coalition formation process continues until time T, and then stops
(ii) The coalition formation process continues until a winning coalition is formed.

In Case (i), the situation is relatively simple. For n-tuples $(y_1,\ldots,y_n)$, $0 \leq y_i \leq 1$, and SN, we define the product functions

$$P_S(y_1,\ldots,y_n) = \prod_{i\in S} y_i \prod_{i\notin S} (1 - y_i) \tag{2}$$

For a given time t, the probability that the coalition favoring σ be *exactly* S would then be given by (2), where the y_i are given by (1).

This expression will of course depend both on the issue position, σ, and the allotted time, t. It should therefore be integrated with respect to some *a priori* distributions on these two variables.

The circumstances under which winning coalitions will be minimal winning is a rather complicated question. The important thing here is the order in which coalitions form. If we set

$$U_i = \lambda_i \| x_i - \sigma \|^{\alpha} \tag{3}$$

then the probability that player i approves *before* player j is quite simply $U_j/(U_i + U_j)$; more generally, the probability that player i be the last to approve from among the set S is given by $U_i/U(S)$, where

$$U(S) = \sum_{j \in S} U_j \tag{4}$$

For relatively small sets N, this can be used to compute the probabilities of the various coalitions without too much trouble; for large values of n, however, extensive computations would seem to be necessary.

If S is a minimal winning coalition, the probability that S be the *first* winning coalition to form is given by the integral

$$Q(S) = U(S) \int_0^1 t^{U(S)-1} \prod_{j \notin S} [1 - t^{U_j}]\, dt. \tag{5}$$

Unless n is small, this intergral will usually be quite difficult to compute. Since, moreover, this quantity depends on σ, a further integration (with respect to the distribution of σ) will be necessary.

The situation is somewhat complicated by the fact that, *under certain circumstances, the first winning coalition to form need not be a minimal winning coalition.* In that case, (5) must be modified to

$$Q(S) = U(S^*) \int_0^1 t^{U(S)-1} \prod_{j \notin S} [1 - t^{U_j}]\, dt. \tag{6}$$

where S^* is that subset of S, consisting of players j such that $S - \{j\}$ is a losing coalition for every j. (Note $S = S^*$ if S is minimal winning.)

(b) The probability that a given proposal will pass can be computed in terms of multilinear extensions [see *Owen*, 1972]. In fact, for a given n-tuple $(y_1, \ldots, y_n)$, we define

$$F(y_1, \ldots, y_n) = \sum_S P_s(y) \tag{7}$$

where the sum is taken over all winning coalitions S.

The probability that, at time t, the proposal be favored by a winning coalition, is precisely $F(\vec{y})$, where y_i is given by (1). This expression should then be integrated with respect to the distribution of t, the "allotted" time.

(c) A modified power index should be possible, also in terms of multilinear extensions.

In fact, for a given value of σ, we can compute the index $z = (z_1, z_2, \ldots, z_n)$ where

$$z_i = \int_0^1 F_i(y_1, y_2, \ldots, y_n) \frac{dy_i}{dt} dt. \tag{8}$$

where F_i is the partial derivative $\partial F / \partial y_i$, and y_i is, once again, given by (1).

3. Role of the Parameters

It is of interest to study the role and meaning of the several parameters λ_i, α, which appear in this analysis.

Let us first consider the parameters λ_i. Essentially, these parameters can be characterized as measures of the "slowness" of the several players to reach a decision. Apart from personal characteristics, these may be meaningful if we think of each player as representing not one, but many voters (party members). A larger party would normally be slower to come to a decision since more members would have to be persuaded. Moreover, some parties may be less willing to accept coalitional roles than others (e.g., so-called "anti-system" parties). The importance of the λ_i can perhaps best be studied if we disregard the distance factor – i.e., assume all the $\|x_i - \sigma\|$ are equal, or, equivalently, set $\alpha = 0$. The results then correspond to those given by *Owen* [1968].

The parameter α, on the other hand, seems to represent, in some sense, the importance which "ideological distance" has on the coalition formation process. In the limiting case $\alpha = 0$, distances are unimportant, and the analysis (as just mentioned) corresponds to that in *Owen* [1968]. In the other limiting case, $\alpha = \infty$, quite to the contrary, the distance is all-important: For a given σ, the order of approval will be precisely the order of increasing $\|x_i - \sigma\|$, with no random factors affecting the order. The situation will then be similar to that treated by *Owen* [1971]. For $0 < \alpha < \infty$, the situation will be intermediate between these two extremes.

4. An Application of the Model to a Hypothetical Legislature

Let us consider the following parliament. There is a large left-wing party, and an equally large moderately conservative party; there are also a small center party and a small rightwing party. A coalition will be assumed to be winning if its seat share is greater than 50. A winning coalition can consist of either (a) the two large parties or (b) one of the large parties with the two small parties. The situation can perhaps be represented by Table 1.

We assume here that the issue space corresponds to the one-dimensional unit interval, [0, 1], and that σ is uniformly distributed here. We shall consider the three cases, $\alpha = 0$, $\alpha = 1$, and $\alpha = \infty$.

Considering the question of which winning coalition will first form, we find that there are five possible coalitions: $\{1,3\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, and $\{2,3,4\}$. Note, as

Party	Position	Votes	Inertia
i	x_i	p_i	λ_i
1	0.1	40	3
2	0.5	10	1
3	0.7	40	3
4	0.9	10	1

Table 1: Hypothetical Example of a Four Party Legislature

S	$\alpha = 0$	$\alpha = 1$	$\alpha = \infty$
$\{1,3\}$	.0357	.0300	0
$\{1,2,3\}$	.1071	.1363	.5
$\{1,2,4\}$	.3750	.3059	0
$\{1,2,4\}$	.1071	.0716	0
$\{2,3,4\}$	.3750	.4563	.5

Table 2: Probabilities that coalition S will be the First Winning Coalition, for Uniformly Distributed σ and various values of α

i	$\alpha = 0$	$\alpha = 1$	$\alpha = \infty$
1	.3347	.2311	.10
2	.1653	.1787	.30
3	.3347	.3431	.40
4	.1653	.2471	.20

Table 3: Power Indices for Parties in the Hypothetical Legislature of Table 1

per our earlier discussion, that not all of these are minimal winning coalitions. Table 2 gives the probabilities that S be the first winning coalition for the given values of α. As may be seen, the "fewest actor" coalition $\{1,3\}$ forms rather infrequently. This is due to the facts (a) that parties 1 and 3 are relatively far apart on the spectrum and, more important (b) that they both have high inertia.[2]) For $\alpha = 0$ or $\alpha = 1$, the coalitions $\{1,2,4\}$ and $\{2,3,4\}$ are clearly the most frequent. For $\alpha = \infty$, however, $\{1,2,4\}$ becomes impossible (at this level of α only convex (i.e., connected) coalitions can form) and $\{1,2,3\}$ becomes as likely as $\{2,3,4\}$.

We also consider our second question, the impact of our assumptions on each party's power. We find, for these values of α, the power indices defined in Equation (8) are given by Table 3.

[2]) If minimal winning coalitions were equally likely, then the $\{1,3\}$ coalition could be expected to form one third of the time.

At $\alpha = 0$, player 1 is relatively strong. As α increases, however, his isolation weakens him until, at $\alpha = \infty$, he is the weakest of the four players. Player 3 becomes very strong as α increases. The behavior of the index for players 2 and 4 is somewhat more difficult to describe.[3])

It is useful to compare the power scores in Table 3 with the usual power indices, the Banzhaf index and the Shapley-Shubik value. There are 16 possible coalitions. The normalized Banzhaf scores for the parties are .3333, .167, .333, and .1667. There are 24 possible permutations. The Shapley-Shubik values are identical to the Banzhaf scores. As can be seen from Column 1 of Table 3, for the case $\alpha = 0$, the variation in λ did not significantly affect the power scores.

We will not address, in this example, the third question considered in the body of the paper, as it depends too much on the *a priori* distribution of t.

5. Conclusions

We hope to have shown how a straightforward model of coalitional dynamics can be used to model two key elements of real-world coalitional processes – (1) Ideological proximities which make coalitions between proximate actors more likely and (2) differences among parties is the rapidity of the decision process (in deciding on whether or not to join a protocoalition) which may be traced to organizational inertia or differential willingness to be part of any coalition structure. Our model, unlike many others in the literatures requires neither minimal winning coalitions to form nor that the final coalition be a connected one. This will depend upon (1) and (2) above.

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[3]) For the simplest case of majority voting games in a unidimensional space where all actors have equal weight p_i and identical position values, χ_i; if we take i to define proximity to the median voter (i.e., $i = 0$, for the median voter and $i = 1$ for the median voter's left hand and right hand neighbors, etc.), then in an analogue to Shapley-Shubik value similar to though much simpler than the approach in this paper, *Grofman* [1979] has shown that power scores are given by

$$\frac{\binom{(N-1)/2}{(N-1-2i/2}}{2^{(N+1)/2}} \quad \text{for } i \neq 0$$

$$2^{-(N+1)/2} \quad \text{for } i = 0.$$

The power scores imputed by this index have some puzzling features. They are not monotically decreasing in i, nor need the median voter be the most powerful actor.

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