TOWARDS A SCIENCE OF POLITICS: ESSAYS IN HONOR OF DUNCAN BLACK

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The Theory of Committees and Elections: The Legacy of Duncan Black*

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I was introduced to Duncan Black in 1968 by Professor E. W. Kelley. Professor Black was then a visiting faculty member at the University of Chicago where I was a graduate student. My M.A. thesis (published in part as Grofman, 1969) was directly inspired by Professor Black's work and benefited from his comments and corrections.

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I. Introduction: The Pure Theory of Politics

Duncan Black's vision is a grand yet simple one: to develop a "pure science of politics" as a "ramified theory of committees" (Black, 1972:3). Black's aim is to place political science on the same kind of theoretical footing as economics.[1] Underpinning all of his work is the deceptively simple insight of modelling political phenomena "in terms of the preferences of a given set of individuals in relation to a given set of motions, the same motions appearing on the preference schedule of each individual" (Black, 1972:3), where motions can be represented as points on a real line or in an N-dimensional space.

Black's great strength is that he is both synthesizer and pioneer. He rediscovered and reinterpreted for contemporary social science the strikingly modern probabilistic and game theoretic insights of long dead theorists such as Dodgson (Louis Carroll), Borda, and Condorcet (e.g., the paradox of cyclical majorities, the Condorcet criterion, the Borda criterion, optimizing the strategies under the limited vote, results on manipulability of voting schemes, the Condorcet jury theorem), while himself developing such seminal ideas as single-peakedness, the importance of the median voter given ordinal preferences, and equilibrium in a spatial voting game.

Because Black is modest about the originality of his own work,[2] because many of the basic ideas in *The Theory of Committees and Elections* appear so "obvious" in retrospect that it is hard to believe that they have not always been part of the stock of general human knowledge, and because few people have seen *Committee Decisions With Complementary Evaluation* (and fewer still have been able to penetrate its opaque style to appreciate its significance), the magnitude of Black's

[1]"What is sought is a theory analogous to that which has dominated economics, in which the observed behavior of all is found as an equilibrium in which each participant is maximizing some suitably defined criterion given institutional and technological constraints and the behavior of others" (Arrow, 1969: 105; cf. Black, 1950: esp. p. 513).

[2]Since *The Theory of Committees and Elections* (1958) incorporates virtually all of Black (1948a, 1948b, 1948c, 1948d, 1949a, 1949b), few readers go back to look at these earlier works. Furthermore, many readers of Black (1958) have neglected to read the preface carefully. For both of those reasons, many who read Black (1958) may be under the impression that it was Black's familiarity with the work of scholars such as Condorcet which inspired his own investigations into committee decision procedures. Actually, Black did not become aware of the work of any of his predecessors until after the first four of his published articles were completed and the fifth almost so (Black 1949a; 159, n.2). Thus, Black recreated on his own the paradox of cyclical majorities and the Condorcet criterion.

research contribution tends to be underestimated. Black's work on what (after him) has come to be called "the theory of committees and elections" is one of the pillars on which rests the theory of public choice and the "new" political economy.[3]

> With no more than simple arithmetic Duncan Black (1958) suggested effects and properties of various voting schemes, e.g., pairwise voting (exhaustive and not), rank-order voting, and extraordinary majorities. He also clarified and carried forward the analysis of single-member district, multimember district, plurality winner, and proportional representation systems. (Fiorina, 1975:147)

Black's work has had a major impact on the development of the field of public choice. In the first 32 issues of *Public Choice*, Black's work is cited in 20 issues, a total of 31 times. Only Downs (1957) and Buchanan and Tullock (1962) are more frequently cited.[4,5] Indeed, the *Theory of Committees and Elections* is one of the most cited works in modern social science, being cited dozens of times each year in the published economics and political science literature.[6] However, like any classic work, its most important effects are indirect, in terms of the literature which it inspired and which now has an independent life of its own.[7] In the remainder of this paper we shall try to inventory Black's key insights and to provide some indications of how the ideas he introduced or reformulated have been applied and developed by subsequent scholars.

II. Research Contributions of Duncan Black

1. The Paradox of Cyclical Majorities

Roughly 200 years ago Condorcet demonstrated that majority rule need not yield a stable outcome when there are more than two alternatives to be considered. Although periodically rediscovered or reinvented by succeeding generations of scholars, the "paradox of cyclical majorities"[8] was, for all practical

[3]Along with the theory of committees and elections, other central components of public choice and the "new" political economy are the theory of parties and candidates (see esp. Downs, 1957), the theory of public goods (see esp. Olson, 1965), the theory of constitutions (see esp. Buchanan and Tullock, 1962), and the theory of bureaucracy (see esp. Niskanen, 1971). (For a slightly different enumeration of components see Tullock, Appendix 2, Buchanan and Tullock, 1962.)

[4]Buchanan and Tullock (1962) is cited in 27 of the 32 issues, for a total of 47 citations; and Downs (1957) is cited in 26 issues for a total of 52 citations.

[5]The next most frequently cited items are Musgrave (1959), Arrow (1963), Tullock (1967), Olson (1965), Niskanen (1971) and Riker and Ordeshook (1968), in that order.

[6]I am indebted to the library staff of the University of California, Irvine, for tracing citations for me using a computerized bibliographic search process. Even though the bibliography at the end of this paper is quite a lengthy one, there are over one hundred and fifty articles which cite Black's work which are not included in it from the period 1972-1979 alone.

[7]For example, Wilson (1972) proves a theorem about single-peaked preferences but cites Arrow (1963) rather than Black. Similarly, Hosomatsu (1974) discusses the Borda rule, citing a secondary source, Young (1974), which cites Black (1958), rather than citing Black (or Borda) directly.

[8]We prefer this name as being more precise than the "paradox of voting." This is also the name preferred by Black (1958:46). (Cf. Fishburn, 1974c.)

purposes, unknown to modern students of democratic theory until called to their attention by Duncan Black (see esp. Black 1948a).[9] In the *Theory of Committees and Elections*, Black demonstrated that the "paradox" was not just a mathematical curiosity, but rather was connected to important political issues such as manipulability of voting schemes (1958: 44; see also 1948a: 29) and the absence of strong similarity of citizen preferences structures (Black, 1958:10-14).

> Although Black was not the first to discover this phenomenon, his work is the foundation of all subsequent research on the problem. The investigations in this field of his principal predecessors, Condorcet and Lewis Carroll, had made no impact on the intellectual community of their day and had been completely forgotten. Their work is known today only because Black, after discovering the phenomenon himself, discovered his predecessors. (Campbell and Tullock, 1965:853)

In its simplest form, the paradox of cyclical majorities arises when 3 voters rank 3 alternatives in order of reference as follows:

1	2	3
x	z	y
y	x	z
z	y	x

A majority prefer x to y, a majority prefer y to z, and a majority prefer z to x. For each alternative, another is preferred to it by some majority.

Three important questions related to the paradox are discussed by Black (1948a, 1958) and have inspired a vast recent literature.

The first question is "Is the paradox inevitable; i.e., do there exist either preference-aggregation mechanisms (voting schemes) or restrictions on the set of feasible preference orderings such that the absence of a clear-cut social choice can be avoided?" This question we shall return to in subsequent sections of the paper.

The second question considered by Black is "How frequently can we expect the paradox to occur; i.e., how likely is it that there will be a majority winner?" The third question is "How easy is it to detect the paradox from the available evidence on majority rule vote outcomes from which voter preference orderings must be inferred?" Still a fourth question, alluded to but not explicitly discussed in Black (1958:46-50) is "If there is a cycle, how large will it be, in particular how many elements will be there in what has come to be called the top cycle?"[10]

[9]Prior at least to the publication of Black (1948a) it seems very hard to justify the claim in Arrow (1963:93) that the paradox of cyclical majorities was well known. The paradox was rediscovered by Huntington (1938), but this work had no discernible impact on subsequent research and also did not connect the problem to issues in democratic theory. Certainly, in political science, the then standard texts on democratic theory and political philosophy make no mention of the paradox. We are aware of only two 20th-century, pre-WW II references to it other than Huntington (1938): the 1907 reprint of Nanson (1882) and Hoag and Hallett (1926). Furthermore, Riker (1965:43) has asserted that as far as he knew "there was no handbook of parliamentary law that mentions the cyclical majority." The present author is a professional parliamentarian familiar with well over a dozen parliamentary manuals and has no evidence to contradict Riker's assertion.

[10]Other names for the top cycle are the Condorcet set (Good, 1971) and the GOCHA Set (Schwartz, 1972).

To the second of these questions, the likelihood of the paradox, Black provides an answer only for the very special case of three voters and three alternatives for what has come to be called (Fishburn, 1974a) the "impartial" culture, i.e., a committee in which all strong preference orderings are equally likely. Black recognized full well, however, the importance of this question and conjectures (Black, 1958:51) that "if the general series could be derived it would almost certainly show that for a committee with a given number of members, the proportion of cases in which there is no unique majority decision increases rapidly with an increase in the number of motions."[11] This conjecture squares well with subsequent results from simulations and numerical approximation formulae. In particular, for an infinitely large committee, "the probability of the paradox occurring approaches one as the number of alternatives increases" (Niemi and Weisberg, 1968:322).

The literature estimating the probable likelihood of occurence of the paradox is extensive. Gilbaud (1952) appears to have been the first to give an exact result for the limit as committee size approaches infinity for the three-alternative case. Campbell and Tullock (1965) are the first to provide extensive simulation results, with further results given in Klahr (1966). Sophisticated approximation and exact calculation methods are offered in Campbell and Tullock (1966), Garmen and Kamien (1968), Niemi and Weisberg (1968), DeMeyer and Plott (1970), Pomeranz and Weil (1970), Weisberg and Niemi (1973), Blin (1973), Fishburn (1973), Gehrlein and Fishburn (1976a) and Gillett (1977, 1978). For the case of the "impartial" culture, the general conclusion of all this research can best be summarized as highly discouraging. The paradox is quite likely indeed. For example, "with as few alternatives as six, and virtually independent of committee size, the probability of no majority winner is almost one-third" (Niemi and Weisberg, 1968:322).[12]

Given such pessimistic findings, results for assumptions more realistic than those of the "impartial" culture assume considerable importance. Here, Williamson and Sargent (1967), Gleser (1969), Jamison and Luce (1972), Fishburn (1973, 1974a), Kuga and Nagatani (1974), Buckley (1975), Abrams (1976) and Gerhlein and Fishburn (1976a, 1976b) provide sophisticated probabilistic methods and Niemi (1969) offers intriguing simulation results

[11]In looking back at his early views on this issue, Black (1972:5 emphasis ours) recalls that "It seemed to me that, for the simple majority procedure, it would be possible to calculate the proportion of cycles. When this proved beyond my own limited mathematical attainments, I thought in terms of a solution got experimentally, but this too proved impracticable. *I continued to take for granted that the computation of this fraction would be an indispensable part of any theory of committees.*"

[12]Black (1972:4 emphasis ours) recounts his amazed reaction in 1942 when he first realized the ubiquitousness of the paradox. "In the early months,... I took for granted that with a simple majority in use, the answer, irrespective of the shapes of the preference curves, would be determinate. Later, working out an arithmetical example in which an intransitivity arose, it seemed to me that this must be due to a mistake in the arithmetic. *On finding that the arithmetic was correct and the intransitivity persisted, my stomach revolted in something akin to physical sickness.* Not only was the problem to which I had addressed myself more complicated than I supposed, it was a different kind. The result would be determinate for only certain shapes of preference curves."

suggesting that partial homogeneity of voter preferences considerably reduces the likelihood of the paradox. On the other hand, recent work on vote trading (e.g., Kadane, 1972; Bernholz, 1973, 1974; Schwartz 1975, 1980) and on cycles over multidimensional issue spaces (e.g., McKelvey 1976; McKelvey and Wendell, 1976; Schofield 1978a, 1978b) provides quite general conditions (see esp. Schwartz, 1980) under which a paradox will be unavoidable. Data from university senate elections using the single transferable vote (the Hare system) in Niemi (1970) suggests that high homogeneity may be realized in practice. Elections under the Hare system require voters to rank-order candidates; for the six three-candidate elections studied by Niemi (1970) between 72% and 87% of the voters who provided complete rank-orders had preferences which satisfy the single-peakedness criterion.

To the third of these questions, how easy is it to detect the paradox, Black offers two useful results. The first of these results (Black, 1958:43) we may restate as "under standard amendment procedure (see Figure 1), given sincere voting, the existence of a voting paradox is always revealed if there are as many rounds of voting as there are alternatives less one, i.e. m-1 rounds of voting."[13] Marz, Casstevens and Casstevens (1973) extend this theorem to the general case of binary voting procedures.

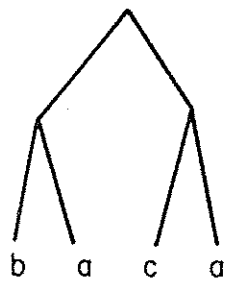

FIGURE 1.1

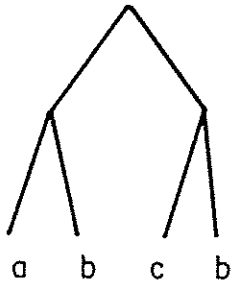

FIGURE 1.2

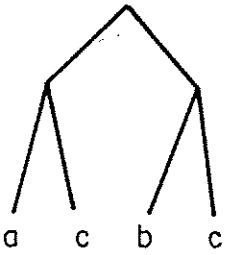

FIGURE 1.3

Standard Amendment Procedure

A second relevant result in Black (1958:43-44) is the theorem that "the voting paradox is always revealed if data is available on all $\frac{m(m-1)}{2}$ paired comparisons." This is a somewhat more powerful result than might be obvious at first sight, since Black (1958:43-44) shows that if there is a majority winner, no single voter has any incentive to vote insincerely in such a complete balloting.

Other analytic results have been demonstrated. For example, Bowen (1972) has shown that in the three-alternative case (where one alternative is the status quo

[13]This theorem is misstated in Murakami (1968:72). In particular, the important first clause is omitted and m is substituted for m-1. With the first clause omitted, the theorem is false, as is noted by Marz, Casstevens and Casstevens (1973) and as an example they present makes quite clear.

and there is one amendment to a main motion being considered), under standard amendment procedure, the paradox can occur only when the final motion (which will be either the original bill or the bill as amended) is defeated.[14] Bowen (1972) and Weisberg and Niemi (1972) provide similar theoretic results for the two amendment case. For standard amendment procedure they note that, i.a., the paradox cannot occur if the alternative winning on the first vote is ultimately adopted.

Several authors have looked at decision-making outcomes in legislatures and in faculty meetings (Riker, 1958, 1965; Weisberg and Niemi, 1972; Bowen, 1972; Lijphart, 1979; Niemi, 1970; Brown and Grofman, 1978; Enelow and Koehler, 1979) to see if occurrences of the paradox could be detected. Bowen (1972) provides probability calculations as to the likelihood of the paradox having occurred in various actual legislative situations, e.g. U.S House of Representatives' voting on the Wheat Act of 1960. Weisberg and Niemi (1972) reanalyze the same data under a slightly different set of assumptions and look at additional bills as well. Riker (1958) looks at the Agricultural Appropriations Act of 1953 and at the School Construction Bill of 1956 (and the Powell Amendment to it—which would have restricted aid to schools which remained segregated) and for the latter bill shows by careful analysis of the historical record that with a the unamended bill, b the bill with the Powell amendment, and c the status quo (i.e. no action) a cycle exists, such that aPb by a vote of 227 to 199, bPc by 229 to 197, and cPa by 217 to 209. Enelow and Koehler (1978) and Enelow (1978, 1979) are other authors who have examined U.S. Congressional role call data to look for sophisticated voting and such contrived paradoxes. Lijphart (1979) concludes that in the Dutch Parliament occurrence of the paradox is quite rare, finding only one involving a 1952 bill on the status of what was then Dutch New Guinea.

Riker (1965) also finds a cyclical majority involving the 1911 Sutherland Amendment to the original text for the proposed 17th Amendment to the U.S. Constitution. According to Riker this was a *contrived* cyclical majority; i.e., legislative preferences appeared to cycle among the unamended bill, the amended bill, and the bill only because some legislators introduced an amendment they did not favor and then voted contrary to their true preferences on its behalf.

Under standard amendment procedure, as Riker (1965) points out, whenever a motion, a, is expected to defeat a motion, c (the status quo), it is open to the supporters of c to defeat a by introducing an amendment, b, such that the supporters of a are divided into two groups, W_1 and W_2, neither commanding a majority, where W_1 members order the alternatives aPcPb and W_2 members order the alternatives bPaPc. This they (supporters of the status quo) can do by voting as if their true preferences were cPbPa. Hence, b will beat a on the first ballot and lose

[14]Clearly, if the amended (unamended) bill passes, then it is preferred to both the unamended (amended) motion and the status quo.

to c, the status quo, on the final ballot, even though a majority actually prefer a to c. (Cf. Dodgson, 1876 reprinted in McGarvey, 1966; Black, 1958, at 233.) Enelow (1979) has developed a probabilistic generalization of Riker's model of sophisticated voting achieving a contrived paradox to take into account imperfect information and legislator uncertainty as to outcomes.

A fourth important question about the paradox of cyclical majorities, how many elements can we expect in the top cycle, is not directly considered by Black. However, Black (1958:46-51) does offer a number of insights on interrelationships between cycles; e.g., "If two (intersecting) cycles have one motion in common, it must be possible to form a cycle which includes all of the motions from both cycles" (1958:Proposition 9, 48) and "If there are three cycles which are nonintersecting, every motion in the first may be able to defeat every motion in the second, and every motion in the second may be able to defeat every motion in the third, and yet every motion in the third cycle may be able to defeat every motion in the first"; i.e., there may be cycles among cycles (1958: Proposition 11, 49-50).

As Bell (1978:122) observes, "it would be comforting to learn... that if a society is unable to produce a Condorcet (majority) winner, the number of alternatives could probably be narrowed to a dominating set of 3 or 4"; i.e., to learn that if there is a cyclical majority, restricting ourselves to an element from this top cycle set would considerably delimit our choices. Unfortunately the results of Bell (1979) show this comfort to be denied us for linear orderings. "In searching for the top cycle set, the smallest subset S of the set of m alternatives for which no member of S fails to win a majority over any non-member, we frequently find that S contains m-1 or m alternatives" (Bell, 1979:122 with some change in notation). Under the standard "impartial" culture assumption, Bell (1979:122) finds that "the number of alternatives in the top cycle is always more likely to be m-1 or m than some number between 3 and m-2 inclusive." For example, "with 15 alternatives the probability of a Condorcet winner is only .39, the probability of a winning cycle including 3, 4,. . ., or 13 alternatives is only .11 while the probability of a cycle including 14 or 15 alternatives is .50" (Bell, 1979:122). The picture is even more bleak if we consider a finite number of voters and an infinite number of alternatives. Under reasonable assumptions, McKelvey (1976) has demonstrated that in this case winning cycles must include all alternatives (see also Schofield, 1978a, 1978b).

2. Single-peaked Preferences and the Median Voter

For Black the ideas of representing motions as points, and of representing preferences as single-peaked utility functions, came together in a flash of insight in February 1942. The story is best told in Black's own words

> The way in which my own work came to be written was that already as a postgraduate student I had got interested in the possibility of the formation of a Pure Science of Politics; and, somewhat later, in the autumn of 1934, I hit on what seemed to me to be the elements of such a science. It was an abstract theory but, to facilitate exposition, it envisaged concrete institutions such as a central government body or a local government body, and attempted to account for the shape they had and also for the shape they might take, if say, for technical reasons, costs and benefits were to alter.
>
> The concepts employed were taken from a theory of the firm that had been developed by a former colleague who, in scores of discussions, had inducted me into his way of looking at this matter (Coase, 1937) and almost equally from the work of the Italian school of writers on Public Finance which at this time was reaching the peak of its achievement. This early work in Politics found itself confronted with a major difficulty—that of making itself intelligible to the reader. The Italians, in the narrower field of Public Finance, had succeeded in giving clear expression to their meaning in only the occasional passage, running perhaps to a few lines or a few pages, or in the occasional diagram, and this despite their very considerable literary abilities.
>
> When the solution to this difficulty did come, *it came in a flash, in the form of a notation in which everything seemed to hang together.* This was in February 1942. A little before then it had been brought to my attention that my main effort during the preceding years had produced no tangible result. I was "firewatching" in case of air raids, late at night in the green drawing-room at Warwick Castle, one of the most magnificent rooms in the whole of England, which even at that time retained much of its former splendour, except that it was strewn with civil-service tables and paraphernalia. *Acting apparently at random, I wrote down a single diagram and saw in a shock of recognition the property of the median optimum.* This could be got by interpreting the diagram I had drawn, in terms of a committee using a simple majority, whose members' preferences, in regard to the motions put forward, could be represented by a set of single-peaked curves. Grant that and the decision of the committee would correspond to the median optimum. I had not previously thought or worked in terms either of motions or of preference curves. I drew two more diagrams and felt sure that I was now able to say the things that I had previously only felt, and could express thoughts that I had been unable to communicate or indeed to formulate properly. Not only so, but I had hit, apparently accidentally, on a technique that would enable a systematic investigation of government to be made along lines that were fairly clearly delineated. Or so the future seemed to stretch before me that night (Black, 1972:4, emphasis ours).

The idea of single-peakedness can be defined in a number of different ways (see, e.g., Arrow, 1963:77; Pattanaik, 1971:73; Fishburn, 1972:95), but we shall follow the familiar and intuitively interpretable graphical interpretation in Black (1948a, 1958), which may be paraphrased as follows: "A set of preference schedules shall be said to be single-peaked if there exists an ordering of the

alternatives such that with respect to this ordering the preference schedules of all committee members could be graphed as single peaked curves, where a single peaked curve is one which changes its direction at most once, from up to down."

There are several points to be mentioned. First, single-peakedness is a property of sets of preference orderings, not merely of individual voter preferences. For a set of preferences schedules to be single-peaked, it is not sufficient that all preferences orderings can be represented as single-peaked curves; they must all be representable as single-peaked curves with respect to the same underlying ordering of alternatives. Second, single-peakedness has a "natural" interpretation in political terms: the closer a motion is to a voter's preferred position on an issue, the more desirable to the voter is that motion. Third, single-peakedness can be interpreted in terms of "similarity" of voter preferences; although voters need not agree on what alternative is most to be preferred, for preference orderings to be single-peaked all voters must agree on the relative locations of the alternatives (including the status quo) in the issue-space. Fourth, single-peakedness can be naturally generalized to multi-dimensional issue-spaces (Black and Newing, 1951).

Most scholars would agree that the two most important contributions made by Black are the introduction of the idea of single-peakedness and the well-known theorem (Black, 1958:14-18) that, for single-peaked preferences, there exists a unique alternative capable of receiving a majority in pairwise competition against other alternatives, and this majority winner will be the most preferred alternative of the median voter.[15] Hence, for single-peaked preferences there will always be a stable majority choice and the preferences of the median voter will be decisive.[16] Furthermore, as Black (1958:19) shows, for single-peaked preferences, majority rule gives rise to a transitive ordering of alternatives.[17]

Black's result on the possibility of stable majority choice given single-peaked preferences has been reformulated by Arrow in his well known "Possibility Theorem for Single-Peaked Preferences" (Arrow: 1963:78-80), in which Arrow demonstrates that, for single-peaked preferences, majority rule establishes a social welfare function which satisfies positive responsiveness, irrelevance of independent alternatives, citizen sovereignty and nondictatorship. Very similar "possibility" theorems have subsequently been proved by other authors (see e.g. Murakami, 1968:124-126; Wilson, 1972). Arrow (1963; see also Inada, 1964) has shown that this result does not actually require single-peakedness with respect to all alternatives; it is enough if we have single-peakedness with respect to every triple of alternatives.

[15]We state the theorem for N, numbers of committee members, odd. Modification of theorem to take into account an even number of members and chairman's tie-breaking vote is relatively trivial. (See Black, 1958:16.)

[16]The importance of the median voter was anticipated by Galton (1907) but only for cases where there was a sum of money to be allocated, with each voter having a preferred allocation.

[17]See note 15.

A number of questions are suggested by Black's single-peakedness/median voter result and its subsequent extensions. First, "Are there other restrictions on preferences sufficient to give rise to (or necessary for) transitive majority choice, or at least for the existence of a majority winner?"[18] Second, "How likely is it that, in fact, voter preferences will satisfy a single-peakedness condition?" Third, "Can Black's median voter result be extended to the multidimensional case?" This third question we shall defer till we consider the work of Black and Newing (1951) in Section 5 below.

The first of these questions is addressed by Black (1958), but only in a very limited way, by considering functions which are single-peaked but with a nonunique maximum (see Black, 1958:25-35). Other authors, inspired by the Black-Arrow possibility result for single-peaked preferences, have exhaustively examined possible restrictions on the feasible preference set.[19] A variety of conditions (well over a dozen) have been proposed. For example, Inada (1964) has looked at single-caveness, which is the inverse of single-peakedness; and at separability of preference over a triple {a, b, c} into two groups, {a} and {b, c}, such that in every preference ordering each alternative in one of the groups is strictly preferred to each alternative in the other. Pattanaik (1971), following Sen (1966:73), has defined "value restrictedness" (VR) as the requirement that in any triple of alternatives there is one alternative such that it is not the worst in any voter's preference ordering of those alternatives (NW), or such that it is not the best in a voter's preference ordering of these alternatives (NB), or such that it is not the medium value in any voter's preference ordering of these alternatives (NM). Pattanaik (1971:91-96) shows that NW is the equivalent of single-peakedness, NB is the equivalent of single-cavedness, and NM the equivalent of Inada's (1964) "separability of preferences into two groups" condition. Each of these conditions is sufficient to avoid a paradox of cyclical majorities (Inada, 1964; Sen, 1966; Pattanaik, 1971). For linear orderings VR is equivalent to Ward's (1964) condition of Latin-Square-lessness. Other conditions include "limited agreement" (LA), proposed by Pattanaik (1971), and "extremal restriction," (ER) proposed by Sen and Pattanaik (1969). For a more detailed treatment, see Fishburn (1972).[20]

While limited agreement and extremal restriction can be given some intuitive meaning in political terms, as can Pattanaik's NM condition (see Pattanaik, 1971:96), single-peakedness (NW) is the most intuitively meaningful of the various domain restriction conditions that have been proposed, although there are

[18]We shall neglect results which involve weakenings of transitivity such as quasitransitivity (see e.g., Schwartz, 1971; Mueller, 1979: 190-191) or contexts where lotteries on alternatives are considered to be feasible alternatives (see, e.g., Fishburn, 1972; Zeckhauser, 1969; Shepsle, 1970).

[19]More technically correct would be to say that authors have looked for restrictions on sets of preference relations over triples of social alternatives.

[20]For definitions of these conditions see Pattanaik (1971:72-75). Limited agreement is a weaker version of a condition called Restriction of Taboo Preference in Inada (1969), while "extremal restriction" is the union of a number of different conditions also first proposed by Inada (1969).

cases in which its complement, single-cavedness (NB), appears to have a useful practical application.[21]

Turning to the second question, the practical likelihood of obtaining single-peaked preference schedules, as Pattanaik (1971:94-95) notes, "single-peakedness (NW) will be satisfied if there exists an agreed-on objective ordering of the alternatives, based on the extent to which they possess a certain attribute. . . (which) constitutes the criterion of individual evaluation." Such will be the case, of course, if there is a clear-cut left-right political dimension on which motions (or political parties) can be judged. Hence, Black's median voter result is directly relevant to modelling political party competition, i.e., if voters prefer the candidate closer to their own ideal point (on a left-right dimension) then the candidate who adopts a position closer to the median preference than his opposite wins the election. Hence, in a two-candidate competition, we would expect both candidates to try to find the median, since the candidate who adopts any other policy loses.[22] However, Black's work deals only with ordinal preferences, and the results in the spatial modeling literature on convergence of two-party politics to the preferences of the median voter rest on Euclidean or other distance metrics.

Single-peakedness can also be related to ideas that have been proposed in the psychological and sociological literature on scaling models. (See e.g., Weisberg, 1972, 1974.) In particular, single-peaked functions are the foundation underlying unfolding theory (Coombs, 1950, 1964/1976) in which an algorithm is provided to measure the variables underlying preferential choice. Applications of unfolding theory have included the study of perception in infants (Thomas, 1973) and the study of preferences for family size (Coombs, Coombs and McClelland, 1975). A wide variety of other applications of single-peaked functions are discussed in Coombs and Avrunin (1977; see also Scott, 1977).[23] Coombs and Avrunin also provide a psychological justification to single-peaked preferences over two dimensions in terms of an approach-avoidance (pain-pleasure) model. However, many of the psychological applications of single-peakedness deal with individual preferences rather than with sets of preference schedules which are collectively single-peaked.

Since, for a linear ordering, single-peakedness is a form of unidimensionality, it would be natural to suppose that single-peakedness was equivalent to Guttman

[21]Pattanaik (1971:95-96) notes that in certain cases involving single criteria of evaluation, NB may be satisified. "Suppose there are three alternatives regarding the organization of a particular industry; complete nationalization; a mixed form of organization, with public and private enterprises operating side by side; and production through private enterprises only. It is possible that every individual gives the "best" value to one of the extremes—complete nationalization or complete private ownership—and nobody gives the "best" value to the mixed form of organization. This implies a purist attitude for all concerned individuals since by moving either side of his least preferred alternative, in the agreed on objective ordering, every concerned individual reaches more and more preferred alternatives." See also Plott (1976).

[22]This result need qualifying in the multidimensional case or if voters are permitted abstention. (See, e.g., Davis, DeGroot and Hinich, 1972; Riker and Ordeshook, 1973; Enelow and Hinich, 1979.)

[23]They note, for example (1977:217) that "one-dimensional single-peaked functions. . . characterize the acceptance-rejection behavior of rats toward various concentrations of sucrose and NaCl solutions." My colleague A. Wuffle, has suggested that such results may be relevant to the revealed preferences of some economists for hot and sour soup. Empirical research on this topic was planned for the 1980 Public Choice Meeting in San Francisco.

scalability (Stouffer et al, 1950). If this equivalent could be demonstrated, then theorems on single-peakedness and equilibrium in voting outcomes would have been shown to be applicable to the legislative or judicial context where analysis of roll call or other voting data has uncovered sizeable subsets of Guttman scalable issues (MacRae, 1970; Rhode and Spaeth, 1976). Unfortunately, Niemi and Weisberg (1974) have shown this equivalence to be false. They show that Guttman scalability does not imply single-peaked preference either for single alternative response tasks, where an individual is asked to respond favorably or unfavorably to each of a series of alternatives, or for paired alternative response tasks, where the individual responds to a pair of alternatives by indicating which he prefers. Elsewhere (Weisberg and Niemi, 1971, Theorem 3, p. 20, Theorem 4, p. 24A), they have shown that Guttman scalability is equivalent to single-peaked preferences only for the highly restrictive case where (a) all possible paired comparisons are made, and (b) all preferences are both single-peaked and symmetric and strictly monotonic. They have further demonstrated that, in general, information about individual votes on a series of items found to be Guttman scalable does not yield the location of voters' ideal points.

Grofman (1976b) has shown that if a set of preferences is single-peaked, symmetric, and linearly ordered along some unidimensional continuum, then the scale patterns generated by all paired comparisons from alternatives along that continuum vs. some one fixed alternative from the continuum will be Guttman scalable when the pairwise choices are left-right ordered according to each alternative's position along the continuum, and the polarity of all columns involving choices located to the left of the one fixed alternative is reversed.[24] This result suggests the following possibility: isolate the set of pairwise voting choices which have some particular element common to all. If decisions over this set are not Guttman scalable, then (assuming sincere voting) the underlying references for alternatives in the set cannot be single-peaked.[25] For this special case, single peakedness implies Guttman scalability. Unfortunately, as is demonstrated in Weisberg and Niemi (1971), the converse is not true.

There are only two direct studies of which we are aware that directly examine the proportion of single-peaked preferences over a set of politically relevant choices. The first of these, Niemi (1970), we have already alluded to. For six three-candidate faculty elections, Niemi finds an average of 78.3% of all complete orderings to be single-peaked. The second study, Grofman and Hamilton (1977), looks at experimentally generated data on verdict choice for four levels of severity of punishment and verdict option. Grofman and Hamilton find that well over

[24]The result as originally stated in Grofman (1976b) is wrong. The symmetry and monotonicity conditions were omitted from the assumptions needed to obtain the result; without them the theorem is false. I am obliged to my colleague, A. Wuffle, for calling this point to my attention. In its correct form the result is a trivial corollary to a theorem in Weisberg and Niemi (1971).

[25]Since in sincere majority pairwise voting a Condorcet choice if one exists will (by definition) defeat all other alternatives, and single-peakedness guarantees the existence of a Condorcet winner (Black, 1958), such a winner whenever introduced into the balloting under standard amendment procedure will remain an alternative in each succeeding pair.

90% of their subjects (college students) ordered verdicts in terms of proximity to that verdict which was considered fairest, giving rise to preferences which were, with only two exceptions (N=24), single-peaked.[26]

We should also recall that not all orderings, or even all triples of orderings, need to be single-peaked in order to generate high probabilities that a Condorcet winner will exist. For unidimensional issues, even 60 percent of all orderings single-peaked gives rise to a quite high probability that the paradox of cyclical majorities will be avoided (see Niemi, 1969).

3. Criteria for Majority Choice and Choice of a Voting Scheme

The problem which puzzled Borda, Condorcet, Laplace, Carroll, and Black was that involved in finding a system of voting which would lead to a choice which could reasonably be regarded as the genuine will of a majority of a group. To people who have not looked into the problem, this seems a foolish inquiry; it seems obvious that a majority is a majority and that is that. In reality the problem is a most difficult one. Even if we accept Black's (1958) dictum that the "candidate who ought to be elected is the one who stands the highest on the average on the committee members' schedule of preferences," where we have more than two alternatives it is not clear what is meant by *the* majority choice. Just as there are several kinds of averages which coincide only in special cases, so, too, are there various reasonable ways of aggregating (or averaging) individual choices.

The most important criterion for majority choice is the Condorcet criterion, named in honor of the Marquis de Condorcet (1743-1794), who was apparently the first to propose it. (See Black, 1958: 166-176; Condorcet 1785.) The Condorcet criterion is deceptive in its simplicity—it is merely that any alternative proposed which is preferred by a majority to each and every other alternative ought to be selected. While such an alternative does not always exist, when it does exist it is unique.

While we agree with Black that there may be more than one "satisfactory" answer to the question of which single candidate ought to be elected in a majority election, we find, as does he, that the Condorcet choice is the single best contender for that honor.[27] If, however, no alternative exists which satisfies the Condorcet

[26]Grofman in other unpublished research (1977) has reanalyzed mock juror verdict choice data in Vidmar (1972) to show that the assumption that juror verdict preferences were single-peaked (a) subsumes the specific hypothesis tested by Vidmar as a special case and (b) provides a better fit to the Vidmar data than an alternative model proposed by Larntz (1975).

[27]"The considerations in favor of the Condorcet criterion are. . . that it is one way of defining which candidate stands highest on the average on the electors' schedules; that it ensures that if one candidate would be able to defeat each of the others in a vote then he will be elected; and that it appeals, perhaps via mathematical symmetry, to our sense of justice. The reasons may not seem overwhelmingly convincing, but we are moving in a region where all considerations are tenuous and fine-spun; and the claims of the Condorcet criterion to rightness seem to us much stronger than those of any other" (Black, 1958). One other important reason for regarding the Condorcet winner as the appropriate outcome of a majoritarian decision process is that the Condorcet winner corresponds to the core of a majority voting game when that core is a singleton. Thus the Condorcet winner corresponds to what is probably the most important solution concept in N-person game theory. (An alternative x_i is said to (majority) dominate an alternative x_j if a majority of voters prefer x_i to x_j. The *core* is the set of alternatives which is not dominated by any other alternative.)

criterion, then we must look to some other criterion or set of criteria for guidance. One such supplementary criterion is the Borda criterion, which may also be considered a rival since it does not always yield the same alternative as the Condorcet criterion when a Cordorcet choice exists.

The Borda rule, named for Jean-Charles de Borda (1733-1799), assigns to each alternative for each committee member one point for each alternative to which it is preferred by that committee member (see Borda, 1781; Black, 1958; Young, 1974); that alternative with the highest number of points, summed over all committee members, is, according to the Borda rule, that which ought to be selected. In short, the Borda count gives the total number of votes each alternative would get if placed in turn against each of the other alternatives. A major advantage of the Borda rule is that it takes into account the overall position of alternatives on the preference scales of each of the committee members. It gives a measure of central tendency. Consider a committee with preference schedules xyqz, xyqz, xyzq, zyqx, zyqx. X satisfies the Condorcet criterion, yet there are good grounds for selecting y. X is either a first choice or a last choice; y is the uniform second choice of all committee members. The Borda count is x, 9; y, 10; z, 6; and q, 5. Thus, the Borda rule would select y. The Borda rule satisfies the positionalist criterion, which can perhaps be best expressed by Fishburn's (1974d) Permuted Dominance condition (which could more appropriately be called Positional Dominance) which says that if x has more first place votes than y, more first and second place votes than y, more first, second and third place votes than y and so on, then y is not in the choice set. This condition is incompatible with the Condorcet criterion as shown by the following example:

1. y x a b c
2. x a c b y
3. y x b c a
4. c y x a b
5. x b a y c

y is the Condorcet winner but is excluded by the PD condition (see also Gardenfors, 1973).[28]

Black's own view (1958:661) is that "the Condorcet criterion should first be used to pick out the majority candidate if there is one; and if no majority candidate exists, that candidate should be chosen who has the highest Borda count." Most authors have concurred with Black that, when there is a Condorcet winner, then that is the alternative which should be chosen. (See e.g., Sen, 1966; Pattanaik, 1968; Fishburn 1970; but compare Fishburn 1971, esp. pp. 136-138 and Batra and

[28]I'm indebted to Jeffrey Richelson (personal communication, January 30, 1980) for calling this point, and the example with it, to my attention.

Yet another reason for regarding the Condorcet choice as important is that it is an accurate predictor of committee choice independent of the nature of the parliamentary features which structure that choice (e.g., the sequence of vote). I am indebted to Charles Plott (personal communication, February 5, 1980) for calling this point to my attention.

Pattanaik, 1972.) However the question of what to do when no Condorcet winner exists continues to generate controversy.

In addition to Black's suggestion that the Borda rule be used to decide cases where no Condorcet winner exists, a variety of other procedures have been proposed. For example, a procedure proposed by Lewis Carroll (C. L. Dodgson) requires us to choose that element which would become maximal with the fewest changes to existing preference orderings (Dodgson, 1876; see Black 1958: 222-233); while Copeland (1951) has proposed that each alternative be given a score equal to the number of y such that xPy minus the number of z such that zPx, and that the alternative with the highest such score be chosen.

Research on criteria of choice in the absence of a Condorcet winner has taken several different directions. One such direction is the attempt to specify criteria on the basis of which procedures such as those proposed by Borda, Dodgson, and Copeland can be compared and judged. A second and closely related line of research involves development of natural "extensions" of the Condorcet criterion to specify choice (usually from among the elements of the top cycle) when there is no Condorcet winner.[29] (See e.g., Goodman and Markowitz, 1952; Ward, 1961; Lady, 1969; Taylor, 1968; Good, 1971; Grofman, 1972; Schwartz, 1972; Smith, 1975; Black, 1976; Fine and Fine, 1974; Gardenfors, 1973; Bowman and Colantoni, 1973; Campbell, 1976; Young, 1974, 1975, 1977; Fishburn, 1974d; Gardner, 1977; Deb, 1977; Packard, 1977; Richelson, 1975, 1976, 1978, 1980a, 1980b; Young and Levenglick, 1978; Miller, 1979.[30])

That two different and both seemingly "fair" voting procedures may not yield the same outcome, and that even two applications of the same voting scheme may not yield identical results when the order in which alternatives are posed is changed are readily demonstrable. It is easy to show that many voting schemes (including most of those in common use) fail to satisfy the Condorcet criterion, when committee members vote sincerely.[31] Black (1958) provides examples which demonstrate this for a number of procedures (even when preferences are single-peaked) including the alternative vote, the Borda rule, and the Hare System as applied to a single-member constituency. We shall provide such examples for the successive procedure (Farquharson, 1969), plurality, and lowest candidate out runoffs (LCOR).[32]

[29]Condorcet's own views on which alternative ought to be selected from among the top cycle set are discussed in Black (1958:174). Although Condorcet's discussion is rather unclear, with at least three interpretations possible, Black (1958:175) finds the best interpretation of his recommendations to be: "discard all candidates except those with the minimum number of majorities against them, and then deem the largest size of minority to be a majority, and so on until one candidate had only an actual or deemed majority against each of the others."

[30]For rather different approaches, see Ferejohn and Grether (1977) and Bowman and Colantoni (1973).

[31]By *sincere voting* we mean as follows: Call the top outcome of a subset that outcome highest on a voter's scale. A voter votes sincerely if he chooses the subset with the highest ranked top. If the tops are equal, he chooses the subset with the highest ranked second-to-top element, and so on (Farquharson, 1969).

[32]Interestingly enough this scheme, despite its common use, is not mentioned in Robert's Rules of Order, which prescribes (1951:276-277) for elections the same procedure as for amendments (i.e., what we've referred to as standard amendment procedure) save that there is no final vote against the status quo.

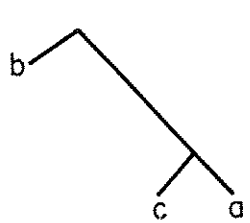

FIGURE 2.1

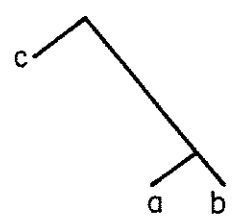

FIGURE 2.2

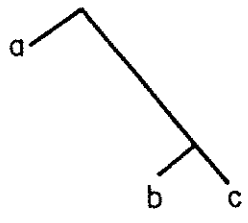

FIGURE 2.3

Successive Procedure

Consider a three-member committee with preferences bac, abc, and cab voting under the successive procedure sketched in Figure 2. Although a is the Condorcet choice, b will be selected, since voters 1 and 3 on the first round of balloting will choose the right-hand subset because that subset {b, c} contains each one's top.

If we look at plurality voting (Figure 3) and LCOR (Figure 4), we may readily establish that neither of these procedures guarantees the selection of a Condorcet choice. Let us assume a committee with five members with preferences abc, abc, cba, cba, and bca. Sincere voting under LCOR leads to the selection of alternative c, even though b is the Condorcet winner. For plurality voting imagine three voting blocs—cba, abc, and bac,—the first of which is the largest but not larger than the combined voting strength of the other two blocs. Under the

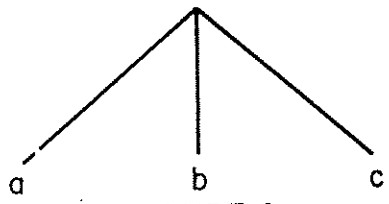

FIGURE 3

Plurality Procedure

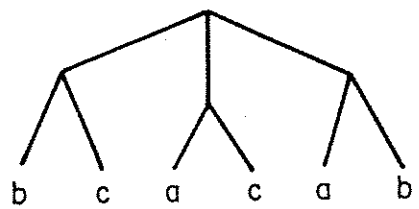

FIGURE 4

Lowest Candidate Out Runoffs (LCOR) Procedure

plurality voting procedure of Figure 3, alternative c will be chosen even though b is the Condorcet choice. This type of outcome is not simply of "theoretical" interest. For example, in the three way 1970 New York Senatorial Race, which pitted two liberals, one a Republican and one a Democrat (Charles Goodell and Richard Otinger) against a Conservative Party candidate (James Buckley), both liberals lost though their combined total was over 60%. In a paired contest, it is likely that either liberal *alone* would have received a majority against the conservative candidate. Sincere voting by many New York voters led a majority of them to a situation in which their least desired choice was victorious.

In contrast to these negative findings, there are two useful positive results due to Black. First, given sincere voting, standard amendment procedure (see Figure 1) always gives to rise to a Condorcet winner if one exists (1958:24); and second, when preferences are single-peaked, sincere voting under the exhaustive voting procedure[33] always selects the Condorcet winner (1958:71). (We shall discuss the consequences of allowing for the possibility of insincere voting in the next section.) In addition, Black (1958:177) offers an example due to Condorcet (1785) for which it can be demonstrated that "no rule which gives a definite mark for a top place, a definite mark for a second-top place, . . ." will always select the Condorcet winner.

A third direction of recent research has been to consider the extent of which different procedures yield the same choice; i.e., how much practical difference would it make if one procedure rather than another were to be used. Of special concern have been the properties of various simple procedures which have been (or might be) used in real-life voting situations and, in particular, the likelihood that a given procedure would select the Condorcet winner if one existed, or that it would select the Borda choice.

Peter Fishburn in singly authored work and in collaboration with William Gerhlein has extensively investigated concordance as to outcome among a number of different voting methods (including both ranked and nonranked schemes) and between different criteria of choice (Fishburn, 1974b; Fishburn and Gerhlein, 1976a 1976b, 1977b; Gerhlein and Fishburn, 1978).[34] Fishburn and Gerhlein (1976b) find the Borda rule unique among positional scoring methods in its likelihood of choosing a Condorcet winner when such exists.[35] Of the various non-

[33]"Exhaustive voting is a process of elimination. If there are, say, six candidates for a post, each member of the committee to make the appointment is given 5 votes, one less than the number of candidates, and at the first round of voting that candidate is eliminated who receives the lowest number of votes. At the second round each voter is given 4 votes and another candidate eliminated and so on, until a single candidate has been 'left undefeated' " (Black, 1958:69).

[34]Condorcet (1785, cited in Black, 1958:179) anticipates this finding when he asserts that "although it (the 'Borda rule') is not exempt from those defects which ought to make us discard the ordinary method, nevertheless these defects are far less pronounced; it is even very probable that it would only very rarely lead into error on the true decision concerning the majority of vote."

[35]An example which makes this point for the Borda rule was given above. A general result (due to Condorcet) about the incompatibility of ranked order methods and the Condorcet criterion will be discussed below.

See also Bjurulf (1978), Paris (1975), Joslyn (1976), Ludwin (1976), Colman and Pountney (1978), Chamberlin and Cohen (1979b). A number of writers on proportional representation have looked at the influence on voting systems by recalculating election outcomes as if they had been conducted under some other electoral system than that actually in use; however, such calculations have usually been of an ad hoc descriptive sort. (See e.g. Lakeman, 1974; Butler, 1953.)

ranked voting schemes considered by Fishburn one, approval voting, where each voter may cast as many votes as there are alternatives less one, has been found to have a number of attractive properties, including a high likelihood of selecting the Condorcet winner (Brams and Fishburn, 1978a; Fishburn, 1978; Brams, 1979).[36]

Chamberlin and Cohen (1978, 1979) have shown how ascertaining the degree of concordance between alternative voting schemes can in many cases be viewed as a problem in linear programming, and derive a number of interesting propositions. For example, they construct (1978) a 19-voter example to verify that there can exist a case in which three different decision rules (Condorcet criterion, plurality, and Borda) can give rise to distinct outcomes (see also Gillett, 1980).

4. Deliberate Distortions of Sincere Preferences and Other Forms of Manipulation

When it was pointed out by Borda that under his voting method it would sometimes be advantageous for committee members to conceal their true preferences in order to increase the likelihood of obtaining outcomes more to their liking, Borda replied, "My scheme is only intended for honest men" (cited in Black, 1958:182). Dodgson, on the other hand, rejected a form of cumulative voting as unsatisfactory because he felt that voters would behave strategically and would assign all their votes to the candidate whom they most favored. (See Black, 1958:218.) Dodgson (1876) also anticipated Riker's (1965) formulation of a contrived cyclical majority arising through insincere voting under standard amendment procedure and comments that this "makes an election more of a game of skill than a real test of the wishes of the electors" (Dodgson, 1876; reprinted in Black, 1958; 232-233).[37]

As noted previously, a number of authors have looked to see if they could find evidence in actual legislative voting of such contrived paradoxes brought about via

[36]See also Brams and Fishburn (1978b), Fishburn (1979), and Merrill (1978)

[37]For such contests Dodgson offers some useful advice:

> In any division taken on a pair of issues neither of which you desire, vote against the most popular. There *may* be some one issue which, if all votes according to their true opinion, would beat every other issue when paired against it separately; but by following this rule, you *may* succeed in getting it beat *once*, and so prevent its having a clear victory, by introducing a cyclical majority. And this will give, to the issue you desire, a chance it would not otherwise have (Dodgson, 1876, reprinted in Black, 1958:233).

To decrease the likelihood of such contrived cyclical majorities Dodgson suggests that, prior to the voting, each voter be required to publicly rank-order the alternatives. He would not, however, bar committee members from voting insincerely with respect to these announced preferences. Dodgson's blend of idealism and cynicism as to committee members' capacity for duplicity is worth reproducing in full.

> The advantages of having the preliminary voting taken on paper and not openly are, first, that each elector, not knowing exactly how the others are voting, has less inducement to vote contrary to his real opinion, so that a more trustworthy estimate is arrived at of the real opinion of the body of electors, and cyclical majorities are less likely to occur, than with open voting; and secondly, that if cyclical majorities do *not* occur in this process, they cannot occur in the formal voting except by some one or more of the electors giving votes inconsistent with their written opinions, and I *think it desirable that in such a case the body of electors should know who they are that have so voted*—a result which this method would secure.
>
> I do not suppose that anyone voting would be so unwilling to have it known that he has so voted that this publicity would *prevent* an artificial cyclical majority—*for I am sure that those who do so believe it an honorable course to take, and have no motive for desiring concealment*—but I think it would increase the sense of the responsibility incurred by those who thus exercise their right of voting, and so make its occurrence less likely. (Dodgson, 1876; reprinted in Black, 1958:233; see also *Ibid.* pp. 237-238.)

strategic voting (e.g., Riker, 1958, 1965; Bowen, 1972; Weisberg and Niemi, 1972; Lijphart, 1979; see also McCrone, 1977).[38]

There are five basic ways in which voting outcomes may be manipulated (Grofman, 1969; Margolis, 1961): (a) by insincere voting; (b) (for some multi-ballot procedures) by specification of the order in which alternatives are voted upon; (c) by making additions or deletions to the alternatives to be considered or by constraining the feasible alternative set; (d) by controlling choice of the voting method (in which category we shall include manipulation of the agenda); and (e) for tiered election systems, by specifying the allocation of voters to districts (or legislators to committees). Black considers three of these five techniques. In our discussion we shall report his results and more recent developments.

(a) Insincere voting: Black (1958:44) reformulates Dodgson's insight on the manipulability of standard amendment procedure and shows that under SAP even with single-peaked preference schedules, "it may be open to one or more of the members to bring into existence a decision more favorable to themselves by voting otherwise than in accordance with their schedules of preference."[39] For complete pairwise balloting, Black (1958:43-44) shows, as we have previously noted, that is is impossible "for any member or group of members, by voting contrary to their schedules of preference, to convert any other motion into the majority motion." For the Borda rule, Black (1958:182) mentions, without much discussion, the possibility of manipulation through insincere voting.

Recent literature has made tremendous advances with respect to the issue of nonsincere voting.

First, Farquharson (1969) has defined sophisticated voting in terms of m-arily admissible strategies and shown that, for binary procedures and complete information, (1) sophisticated voting is determinate,[40] and (2) sophisticated voting yields the Condorcet choice if one exists.[41] Since most committee voting is binary (i.e., either standard amendment procedure or the successive procedure),

[38]Another area where the empirical occurence of sophisticated voting has been looked at is referendum voting. See Garriga-Pico (1979).

[39]About the likelihood of strategic voting under standard amendment procedure, Black (1958:45 with some change of sentence ordering, emphasis ours) has this to say: "We know from experience that people do not invariably. . . vote directly in accordance with their schedules of preference; and now we have shown that when the ordinary committee procedure is in use, it may be against their interest to do so. In this case, therefore the theory may sometimes fail to correspond to reality. *But the restriction on its applicability from this source is probably not very significant.*" Given the evidence on legislative voting under the standard amendment procedure we have available (e.g., Riker, 1965; Lijphart, 1975; Bowen, 1972), this conclusion does not seem erroneous.

[40]Binary procedures are those in which the set of outcomes at each round of balloting is partitioned into two subsets, until single outcomes (decisions) are reached. We may depict the results of rounds of balloting as an "outcome tree," where a fork corresponds to a vote and the end of a branch corresponds to an outcome. Figures 1 and 2 specify the set of distinct binary voting procedures for the three alternative cases. Figures 3 and 4 represent ternary procedures. A voting procedure is said to be determinate under sophisticated voting if, for any set of (strong) voters' preference orderings, it results in the selection of a single specifiable outcome. (See Farquharson, 1969; Brams, 1975:76-77.)

[41]The proof of the former result in Farquharson (1969) is in error. See correction in Niemi, McKelvey and Bjurulf (1971:22). For easy to follow proofs of this latter result, see Niemi, McKelvey and Bjurulf (1971:9) and Miller (1977b).

there are very powerful results. Farquharson's (1969) work is connected to important equilibrium notions in N-person game theory. (See Brams, 1975; Dummett and Farquharson, 1961; McKelvey and Niemi, 1978; cf. Majumdar, 1956.) Farquharson's basic ideas have been reformulated and extended in graph-theoretic terms by defining the notion of a "sophisticated equivalent" at a decision node (Niemi, McKelvey , and Bjurulf, 1974; Miller, 1977b; McKelvey and Niemi, 1978). This work offers conceptually elegant insights into committee decision processes.

Second, the phenomena of manipulability of voting schemes through insincere voting has been found to be virtually inescapable. Gibbard (1973) and Satterthwaite (1975) have demonstrated very similar results—which have come to be known as the Gibbard-Satterthwaite theorem. We may paraphrase their findings somewhat loosely as follows: "For any voting scheme which is nontrivial (where triviality has a precise technical meaning which we won't go into here), there are some preference schedules of committee members such that not all committee members will possess a dominant strategy; hence it will be potentially advantageous for one or more committee members to vote insincerely."

The Gibbard (1973) and Satterthwaite (1975) results are rather general ones, applying not simply to the majoritarian voting schemes we have considered so far, but to a far wider class of schemes for preference aggregation including ones where voters' preferences are weighted unequally and/or where special majorities are required.[42] The Gibbard-Satterthwaite Theorem shows that manipulability is similar to dictatorship in the sense that, if we want a collective decision scheme with various "nice" properties, but without restrictions on the domain of admissible preferences, then we may be stuck with a scheme which is on occasion either manipulable or dictatorial (cf. Peleg, 1978; Dutta and Pattanaik, 1978). However, most of the manipulability results have been confined to systems which are "resolute," and this is a rather strong assumption (see Schwartz, 1980).

Just as a natural direction to look for escape from the Arrow Impossibility result was in terms of domain restrictions, so, too, recent work on manipulability has sought to determine if there are schemes which cannot be manipulated through insincere voting when there are restrictions on admissible preferences. The most interesting domain restriction for our purposes is, of course, single-peakedness. Dummett and Farquharson (1961) have shown that majority rule with Borda completion (i.e., the decision process recommended by Black) is strategy-proof if *both* admissible preferences and admissible ballots are required to be single-peaked. Blin and Sattherthwaite (1976) show that neither of these conditions is alone sufficient.[43] As Blin and Satterthwaite (1976:57) note, this

[42]These results have been extended to deal with procedures involving chance mechanisms in Gibbard (1977) and Barbera (1977). Other research which is relevant deals with the search for voting shcemes with "nice" equilibria (those which correspond to sincere voting), e.g., Dutta and Pattanaik (1978) and Peleg (1978). See also Pattanaik (1975), Gardenfors (1976).

[43]Dummett and Farquharson (1961) also provide nonmanipulability results for domain restrictions other than single-peakedness. Other research along these lines is Kalai and Muller (1977) and Blair (1979).

double requirement makes it difficult for a group to realize the potential for strategy-proofness.

> If a group considers a sequence of issues such that members' preferences over the alternatives contained within each issue are certain to be single-peaked, then the requirement for such agreement may be easily met. The group can make a single once and for all decision that requires the casting of single-peaked ballots. If, however, the sequence of issues which the group considers has more variety and preferences cannot be assumed to be single-peaked, then this becomes a very difficult requirement because a once and for all decision is inappropriate. The decision to restrict the set of admissible ballots must be made anew for each issue. But making the restriction of the admissible ballot set itself an issue is self-defeating. By voting strategically on the subsidiary question of whether to restrict or not to restrict the admissible ballot set, individuals may successfully manipulate the group's final decision among the elements of S. The existence of this possibility is a prima-facie violation of the concept of strategy-proofness.

If we must expect that, in general, all voting schemes sometimes offer opportunities for some voters to vote insincerely to their advantage, it makes sense to ask "In practice, are some schemes more likely to generate strategic voting than others?". As Chamberlin and Cohen (1978:1) put it, among the questions one might ask are:

(1) In what proportion of elections is a particular social choice function manipulable?
(2) How many voters must misrepresent their preferences in order to change the outcome?
(3) What proportion of the voters with an interest in a given change must misrepresent their preferences?
(4) How easily can one characterize the strategies necessary for manipulation?

Chamberlin and Cohen (1978) address some of these questions using ballot data from the five-candidate, 11,586 voter, 1976 election for the President of the American Psychological Association, and using preference data reconstructed from thermometer scores assigned by survey respondents to various political figures who were prominent contenders for the 1972 and 1976 Democratic presidential nominations.[44] He looks at four procedures which can be used for single-candidate elections: plurality, the Borda rule, the Hare system, and the Coombs system.[45]

[44]Subdividing his preference data by region and by feasible candidate set, Chamberlin (1978) creates 20 "hypothetical" elections.

[45]The Coombs system is identical to the Hare system (single transferable vote) except that if no candidate receives a majority of the first place votes, the candidate with the *most last place votes* (rather than the candidate with the *fewest first place votes*) is eliminated and the (first place) votes assigned to that candidate are transferred to the next ranking candidate(s). As with the Hare system, the process of eliminating the currently most disliked candidate continues until one candidate has a majority. The Coombs system was first proposed in Coombs (1964).

We might also note that of the four voting methods listed above (plurality, Borda, Coombs, Hare), the Coombs system was the one found by Chamberlin and Cohen (1979) to most often satisfy the Condorcet criterion.

While for plurality, the Borda rule, and the Coombs systems Chamberlin finds the immunity to strategic voting to depend upon which criteria are used to measure degree of manipulability, Chamberlin's overall conclusion (1978:11) is that "The Hare system is much less manipulable than the other three functions." Furthermore, he asserts that he would expect this difference to remain even after more general analysis" (1978:11).[46] With respect to Chamberlin's fourth question, the ability of voters to perceive the strategies they must use in order to manipulate the outcome, Chamberlin (1978:5) asserts that "For plurality voting and the Borda rule, the strategies are straightforward, for the Hare system, they can be quite difficult to determine."[47]

Niemi, McKelvey, and Bjurulf (1974), Fishburn (1978), and Brams and Fishburn (1978, 1979) are other authors who have looked at the "degree" to which particular voting schemes are manipulable (see also Brams and Zagare, 1977). Brams and Fishburn show that approval voting provides no incentives to insincere voting for certain restrictions on voters' preference orders (e.g., dichotomous preferences) and that, in general, approval voting will offer fewer incentives to strategic voting than a host of other schemes (including plurality and second-ballot runoffs). Niemi, McKelvey, and Bjurulf (1974) look at manipulation in terms of voting order effects, and we shall report their results when we discuss that issue below. Before turning to that topic, however, a brief comment on the ethics of "insincerity" seems in order. The Farquharson/Niemi-McKelvey-Bjurulf/Miller result that, for binary voting procedures, (individual) sophisticated voting can guarantee choice of the Condorcet winner (if one exists) suggests that one kind of insincerity (i.e., sophisticated voting) can be far from evil in its effects. In the words of Niemi, McKelvey, and Bjurulf (1974:10, emphasis ours),

> Most committee procedures are binary and hence come under the scope of the above arguments. It follows that although it is sometimes possible to frustrate the will of a majority through the adoption of appropriate voting procedures, *this is not possible when sophisticated voting is operative.*
>
> The implications of this result are important. In order to be able to use a sophisticated strategy, a voter needs access to information on the preferences of others. This implies that lack of pre-vote communication and withholding of information may lead to socially less desirable outcomes. *Furthermore, the idea that there seems to be something inherently wrong about. . . misrepresenting one's preferences to advance one's personal gain seems to be wrong under binary procedures.* Rather, in such cases, sophisticated voting leads to socially more preferred outcomes.[48]

[46]Recall that Chamberlin (1978) is dealing with the Hare system only in the context of single-candidate elections.

[47]For details see Chamberlin (1978:5-7). For more detailed discussions of manipulation possibilities under the Borda rule see Fishburn (1974c, 1974e), Gardner (1977), Ludwin (1978).

[48]Very similar points are made in Miller (1977b:80). Of course, some voting contra to one's sincere preference, e.g., certain patterns of vote trades may lead to Pareto inferior outcomes. See e.g., Riker and Brams (1973), Schwartz (1977), Enelow and Koehler (1977).

(b) Voting order: Many multiballot procedures can be manipulated in terms of the sequence in which alternatives are voted on.[49] Black (1958:40) points out that "for standard amendment procedure, given sincere voting, the later any motion enters the voting, the greater its chance of adoption." For binary procedures, the possibilities for manipulation of voting order are now well determined thanks to the work of Farquharson (1969) and its continuation by Niemi, McKelvey and Bjurulf (1974), and Miller (1973, 1975a, 1977b). The key result is that under both standard amendment procedure and the successive procedure (see Figures 1 and 2), when no Condorcet winner exists, it is advantageous (or at least no worse) to have one's first choice voted on later rather than earlier when (all) voters vote sincerely, but earlier rather than later when voters are sophisticated. (For proofs of this result, see Miller, 1975a, 1977b; Niemi, McKelvey and Bjurulf, 1974.) Niemi, McKelvey and Bjurulf (1974:14, Tables 2 and 3) also have recommendations as to *when* in the balloting a voter may prefer to have certain nonpreferred alternatives voted on.

Because sincere voting satisfies the Condorcet criterion under standard amendment procedure but not under the successive procedure, Niemi, McKelvey and Bjurulf (1974: 13-14, emphasis ours) conclude that:

> (T)he successive voting procedure seems more vulnerable to strategic manipulation than does the amendment procedure. Under the amendment procedure, the order of voting is a factor only when there are cyclical majorities, a circumstance which may not occur very often, and which will probably be troublesome under any voting system. The successive procedure is vulnerable to voting order effect even when voting is sincere, and there is a majority winner. Unless the majority winner also has a majority of first place preferences, it may lose. *Judged by this criterion alone, then, we might conclude the standard amendment procedure is superior to the successive procedure, if our goal is to devise systems of voting which best reflect the underlying preferences of voters.*

(c) Additions or deletions to the set of feasible alternatives: This is a form of voting manipulation which, as far as we are aware, Black nowhere discusses. It is easy to show that, in the absence of domain restrictions or some form of strategic voting, outcomes of all voting procedures may be subject to this type of manipulation.[50] We shall provide examples for a number of voting schemes.

Consider a five-member committee voting by LCOR with preferences pxyz, pxyz, zxyp, yzxp, yzxp. LCOR selects, assuming strict adherence to preference schedules, y. If the (irrelevant) alternative p is deleted, then x will be selected. P drew strength away from x, since for four of the five committee members the preferences for x and p were virtually identical. In LCOR, outcomes may be

[49]Interesting empirical discussions of the practical importance of manipulation of voting procedures are found in Epstein (1977), Plott and Levine (1978), and Bjurulf and Niemi (1978).

[50]This result is related to the axiom of revealed preference. See Grofman (1969), Fishburn (1971, 1974c), Schwartz (1981, forthcoming).

manipulated by the introduction of (irrelevant) alternatives which strongly appeal to the same voters as the widely supported front runner(s) whom one wishes to defeat. If this strategy is successful, moderate candidates may never make it to the later stages of balloting in which their widespread acceptability (when first and second choices were no longer in the running) would give them a chance of victory.

Similarly, consider a three member committee voting sincerely by SAP with preferences pxyz, zpxy, and yxzp. If the alternatives are voted upon in the order first p and x, then y, then z; then z will be selected. If p is deleted from the set of alternatives being considered, with the order of voting among the remaining alternatives remaining unchanged, then x will be selected.

The Borda procedure is also open to such manipulations. Consider a five-member committee with preferences yxzp, yxzp, pzyx, zpyx, zyxp. Z is the Condorcet winner and also the Borda winner (x=6, y=9, z=10, p=5). If we delete the majority-dominated alternative p, then the Borda count become x=2, y=7, z=6 and the Borda winner is now y. Fishburn (1974c, 1974e) provides some lovely examples of manipulation of the Borda rules by specification of the set of feasible alternatives, including some in which the collective preference ordering among all remaining alternatives is reversed when one alternative is removed from the feasible set.

Given sincere voting, it is easy for the plurality procedure to construct an example whereby outcomes are manipulated by the introduction or deletion of certain alternatives. Consider the 1968 U.S. Presidential election—Wallace's vote came slightly more from among Nixon-leaning Democrats and Republicans than from among Humphrey-leaning Democrats and Republicans. Thus, even though Wallace drew more heavily from among Democrat identifiers than from among Republican identifiers, the net impact of his candidacy was (contrary to popular opinion) on balance favorable to Humphrey (Converse et al., 1969).

(d) Choice of Voting Method: As we have noted, Black (1958) demonstrated that (for sincere voting) various procedures failed to select the Condorcet winner even when one existed, with some even failing to do so when preferences are single-peaked; and observed, more generally, that choice of the voting method could effect choice of the outcome. As discussed previously, some recent work has dealt with the extent to which different procedures (in practice or in principles) yield different outcomes(see citations above), and a method for verifying conjectures about possible divergences of outcome among different procedures has been developed (Chamberlin and Cohen, 1979a).

Recent work on manipulation of voting outcomes has also dealt with ways of contriving stable majority outcomes by mechanisms such as parliamentary procedures which limit permissible amendments;[51] or institutional structures

[51]For a still very useful (although, unfortunately, quite dated) introduction to U.S. Congressional procedures and the ways in which they have been or might be manipulated, see Froman (1958).

which disaggregate issues so as to rule out the need to choose among all possible issue platforms (for example, a committee system; or agenda control techniques which divide any question into separate issue domains and specifies the order in which these shall be considered,[52] or which simplify choice by combining positions so as to restrict choice to two (or at most a few) competing issue platforms (for example, via mechanisms which (by restricting entry) institutionalize two-party competition) (Shepsle, 1978; Levine and Plott, 1977; Tullock, 1967; Slutsky, 1977; see also Romer and Rosenthal, 1979; Mackay and Weaver, 1979).[53]

We shall return to the question of choice across multiple issue dimensions when we consider the contributions of Black and Newing (1951) in the section below.

(e) Distribution of voters across districts: For a single-member district based legislature, in which legislators seek to follow the majority preferences of their constituents, Margolis (1961) briefly discusses the extent to which the drawing of district boundaries might be manipulated so as to affect the likelihood that a Condorcet winner will exist at the legislative level. As far as we are aware, this issue has never subsequently been pursued by other authors writing in the public choice area. However, closely related questions have been dealt with in the literature on reapportionment in terms of majority representation in the legislature (see Johnston, 1979 for extensive review; cf. Niemi and Deegan, 1978) and in discussions of the implications of a legislature's committee system and committee assignment procedures for legislature outcomes (Shepsle, 1978). Clearly, in representative systems, the allocation of voters to constituencies can have major consequences for election outcomes and subsequent legislative policies.

5. Contributions of Black and Newing to Spatial Modelling

While the notion of viewing political choice in spatial terms is not original to Black, its primary antecedents in the economics literature (Hotelling, 1929; Smithies 1941) are in terms of oligopolitical competition between parties viewed as firms. We may with justice assert that the first full-blown political application of the so-called "spatial" model, i.e., one in which alternative social states are viewed as points in a convex policy space (such as E^n), is Black and Newing (1951).[54]

Black and Newing (1951) present their analysis largely in graphical terms, and the style of presentation is quite confusing, leaving the reader inundated with

[52]When preferences over issue dimensions are not separable, choice among all possible platforms is extremely unlikely to give rise to an equilibrium outcome, even if preferences on each separate issue dimension are single-peaked. (See Plott (1973), McKelvey (1976), Black and Newing (1951), Black (1958:139), and our discussion in section below.)

[53]The above articles represent extremely important lines of research which have only just begun to be explored.

[54]See, however, Bowen (1943).

diagrams and derivatives and rather at a loss as to what general results, if any, have been demonstrated. Nonetheless, the few scholars who have carefully examined this work (see e.g., Plott, 1967) have found it to be a pioneering contribution to our understanding of the conditions for the existence of a stable voting equilibrium in a multidimensional issue space. As Sloss (1973:19) puts it, Black and Newing (1951) present "a very complete and general analysis in geometrical terms for the 3-person case, where alternatives can be represented as points in E^2; and they extend some of their results to the N-person case."

Black and Newing (1951) deal with what they call "complementary valuation," (which would now be called nonseparable preferences) over a two-dimensional choice space. They show that majority voting equilibria require extremely restrictive assumptions as to the nature of the intersection of voters' indifference contours and that if an equilibrium point exists it must be the optimum for at least one individual (Black and Newing, 1951:21-28); but also show (1951:31-49) that when preferences on each issue are single-peaked, a stable local equilibrium can be arrived at via a sequential process which treats choice on one of the issue dimensions as fixed and specifies the choice on the other issue dimension, and which continues in this fashion but switching each time the dimension which is regarded as fixed.[55]

The results in Black and Newing (1951) have been rigorously proved and extended in three important papers. Plott (1967) give conditions which are both necessary and sufficient for the existence of a stable outcome in a class of multi-dimensional voting problems. Those require extreme symmetry; for every voter assigned a preference ordering of one type, another voter must be assigned an ordering of a complementary type. We may restate Plott's necessary and sufficient conditions for the existence of an equilibrium as (1) any equilibrium point must be an optimum for at least one individual, and (2) if the point is an optimum for one and only one individual, then the remaining individuals can be divided into pairs whose contract curves pass through the equilibrium point.

Adapting and extending ideas in Farquharson (1969), Kramer (1973b) shows in N-dimensional space that there exists a "sophisticated" equilibrium when changes from the status quo must be made one issue dimension at a time.[56] Kramer (1973a) shows that, in multi-dimensional issue space, where preferences can be represented by quasi-concave differentiable utility functions, the various domain restrictions sufficient for majority rule (see section 2 above), including single-peakedness, are incompatible with even a very modest degree of heterogeneity of tastes; and for most purposes are probably not significantly less

[55]Black and Newing (1951) also look at the dynamics of convergence (divergence) of voting process in a multidimensional space. McKelvey and Wendell (1976), McKelvey (1976), Schofield (1978), and Kramer (1977) pursue research on related issues. Elegant empirical work on this point is contained in Fiorina and Plott (1978).

[56]"In effect. . . single-peakedness fails if there exists a point at which the marginal rates of substitution of any two voters differ" (Kramer, 1973a).

restrictive than the extreme condition of complete unanimity of individual preferences" (1973a:285).[57]

Buchanan (1968:110-114) notes that when issues are considered one at a time, the "generalized" median (i.e., the platform which consists of median positions on each issue) would be chosen. Kadane (1972) demonstrated that for separable preferences, the generalized median is always a member of the Condorcet set. Related questions are considered in recent work on logrolling and vote-trading (see e.g., Buchanan and Tullock, 1962; Bernholz, 1966, 1973, 1975, 1977; Schwartz, 1975, 1977, 1980; Miller, 1975b, 1977a; Sullivan, 1976; Enelow, 1977b; cf. Weingast, 1978), as well as in extensions of Downsian theories of political party competition where "dominant campaign strategies" play the role of equilibrium outcomes. (See e.g., Davis, Hinich, and Ordeshook, 1970.) Other innovative recent work on committee choice over multi-dimensional issue spaces includes Rae and Taylor (1971); Wendell and Thorson (1974); McKelvey and Wendell (1976); McKelvey (1976); Schofield (1977, 1978); and Kramer (1977).

6. Special Majorities, Side Payments

Although he does not use this language or the corresponding notation, the bulk of Black (1958) treats the committee decision problem as if it were a majority (spatial) voting game without side payments. Two extensions suggest themselves: first, a majority voting game with side payments; second, a special majority game without side payments. The first type of extension Black, in effect, considers in Chapter 14 (1958); however, the graphical methods used enable him to get no particularly useful handle on this problem.[58] The second type of extension, requiring special majorities, Black considers in Chapter 13 (1958). As he puts it, "the problem is to investigate the tendency to increased stability of a motion already in force, when there is an increase in the size of a majority that any other motion must get in order to supersede it." (1958:100)[59]

Introducing the notion of elasticity with respect to changing size of majority required to topple the status quo, Black presents some graphical examples to suggest how, as the decision requirements for change move to unanimity, almost any status quo becomes impossible to overturn. This suggests that those who view the status quo as a desirable state of affairs for themselves ought to favor rules which make change difficult. Black's (1958) treatment can usefully be contrasted with the discussion in Buchanan and Tullock (1962) on the optimal number of

[57] Kramer (1973b) does not appear to be aware of the connection between his result and the work of Black and Newing (1951).

[58] This is one of the least satisfying chapters in Black (1958). Black is dimly grappling with the quite difficult problem of collective choice, when there are differing *intensities* of preference and the possibility of side payments. Without a game-theoretic apparatus and the introduction of V-M utilities, little can be said. Black (1966) does, however, provide an interesting reformulation of standard game-theoretic ideas in a purely ordinal utilities framework. For some recent work dealing with N-person voting games, see e.g., Ordeshook, McKelvey, and Winer (1978), Fiorina and Plott (1978).

[59] Kramer (1977) addresses a rather different but still related problem.

individuals required to take collective action. Buchanan and Tullock (1962:83) approach the problem from the Wicksellian view that the further the decision rule is from unanimity the more likely it is that a change from the status quo will be foisted on an individual against his will. They modify this model, however, by incorporating a notion of decision costs such that the more people who need to be persuaded the harder it is to achieve any desired change, concluding that the decision rule is best which offers optimum trade-off between expected benefits (from desired changes) and the expected costs (from preventing undesired ones). (Cf. Rae, 1969; Taylor, 1969.)

Black, addressing the desirability of unanimous vs. nonunanimous agreement procedures in the context of international agreements, opts for unanimity in this quite special case (Black, 1958:151-152). However, Black (1969b) offers a rather strong argument against the reasonableness of a unanimity requirement for committee decision making. (See also Black, 1972.)

The question of the desirability of majoritarian vs. supra-majoritarian procedures has, of course, been of concern to many scholars both within and outside of the public choice tradition (see e.g., Heinberg, 1932; Kendall, 1941; Reimer, 1951; May, 1952; Dahl, 1956; Buchanan and Tullock, 1962; Rae, 1969, 1975; Taylor, 1969; Badger, 1972; Curtis, 1972; Scholfield, 1971, 1972; Mueller, 1979:207-226; Grofman, 1976, 1980; and see our discussion of the Condorcet jury theorem in Section 10 below).

7. Reconstructing Voter Preferences from Ballot Data

Consider a set of majority voting outcomes on *all* pairwise choices among some set of alternatives. If all voter preferences are strongly ordered, can we uniquely reconstruct individual preference schedules? Black (1958:119-120) provides an answer to this question in the negative; generally, a given set of outcomes will be consistent with more than one set of preference schedules. Black (1958:124-125) also shows that, as might be expected, some sets of majority voting outcomes may be impossible to reconcile with any assignment of strongly ordered preferences. The method used in Black (1958, Chapter 15) is the solution of a simultaneous set of linear equations. When ballot-derived information is incomplete (which is true for most voting procedures in that all pairwise comparisons are not known), then simultaneous equations may still be useful in discovering what preference schedules are compatible with the observed results. However, when ballots are few relative to alternatives, the sets of orderings compatible with observed outcomes may be very large (Black 1958:54; cf. Coombs, 1964 for the case of single-peaked preferences).

The problem is made considerably more difficult if sophisticated voting is possible (see McCrone, 1978). Reconstruction of voter preference orderings from ballot information has been attempted by a few authors (see e.g. Riker, 1965; Lijphart, 1975; Brown and Grofman, 1978; Enelow and Koehler, 1979; Enelow,

1979). The linear programing techniques discussed in Chamberlin and Cohen (1979a) appear well suited for such analyses.

8. Social Choice and Social Ordering

While Arrow's theorem can best be thought of as the answer to a question first posed by Bergson (1938) on the existence of social welfare functions,[60] it is also closely related to the work of Condorcet; although of course, Arrow's conditions can be violated and yet a Condorcet winner exist, since a Condorcet winner does not require collective rationality. It was rather unfortunate that the connections between social welfare and majority choice criteria are not addressed in the first edition of *Social Choice and Individual Values,* but Arrow's second edition (1963:93-96) discusses the work of Condorcet, Dodgson, and others in some detail.[61]

Like all other contemporary writers who have discussed the historical roots of the theory of committees and elections (e.g., Riker, 1961; Tullock, Appendix 2 in Buchanan and Tullock, 1962), Arrow (1963:94) fully acknowledges the magnitude and importance of Black's (1958:156-238) contribution to historical scholarship, on which he relies for his own review of pre-20th century research.

> Black has given a history of the theory of social choice, starting with the work of Borda and including that of Condorcet, Caplace, Nanson, and most especially C. L. Dodgson (Lewis Carroll). In regard to the last, he has uncovered some previously unpublished pamphlets in which Dodgson cryptically, although with great acumen, analyzed problems of elections and particularly what he called 'cyclical majorities.' Both Dodgson's work and Black's comments on it and on the circumstances of its origin are extremely worthwhile. Black's excellent history makes superfluous any need for recapitulation here.

Nonetheless, I believe that, with some important exceptions,[62] Arrow (1963) does not go far enough in appreciating the similarity between the problem which vexed Condorcet and his successors (including Black) and the problem which his

[60]This view of the connection between Arrow's work and that of the "old" welfare economics is not accepted by Bergson himself. See Bergson (1954) and rebuttal thereto and discussion and further references in Arrow (1963:103-105).

[61]In the second edition, Arrow (1963:93) remarks that "I must confess to a certain want of diligence in tracking down the historical origins of the theories of social choice. When I first studied the problem and developed the contradictions in the majority rule system, I was sure that this was no original discovery, although I had no explicit reference and sought to express this knowledge by returning to the well known 'paradox of voting.' "

The only citation in Arrow (1951) to the paradox of cyclical majorities is Nanson (1882, reprinted 1907). Arrow's unfamiliarity with works earlier than Nanson is not surprising since (1) Nanson himself was apparently familiar neither with the work of his contemporary, C. L. Dodgson, nor with the scholars of the century previous to his own, and (2) although Arrow is familiar with Black (1948a), the first reference to the names of either Borda or Condorcet in Black's work comes in Black (1949a), which wasn't published until after the draft of Arrow's manuscript had been completed. (*Social Choice and Individual Values* was begun in 1948 and completed in June 1949 (Arrow, 1951: Preface).)

Furthermore, even if Arrow had read Black (1949a), that article only mentions the names of Borda and Condorcet and does not give citations to the writings in which the paradox or the Condorcet criterion are discussed.

[62]See especially Arrow (1963: Chapter 7, esp. 75-80).

theorem was addressing.[63] While it is true that Black and his predecessors were concerned with the issue of political choice, not that of social welfare (Arrow, 1963:80), and while it is true that the social ordering (social welfare function) which Arrow is looking for is a much stronger concept than that of the Condorcet winner or of the core (the Condorcet set), there is an intimate connection between the idea of a social choice set and that of a social ordering. (On this point see e.g., Dummett and Farquharson, 1961; Murakami, 1968.)

9. Proportional Representation

Black's contributions to the literature on proportional representation are threefold:

First, Black (1967, 1970) clarifies the historical roots of the P.R. movement. Black's contributions to historical understanding in this area are of the high level of painstaking scholarship and careful analysis we have come to expect from Black (1958), although they fill in only a part of P.R.'s history. Thus to our loss, we lack a comprehensive historical overview of the theoretical development of P.R. of the sort that Black (1958) so beautifully provides for the logic of single-member elections.

Second, Black (1967, 1969b) offers a clear exposition of the logic of the limited vote and of Charles Dodgson's (Lewis Carroll's) *The Principles of Parliamentary Representation* (1884). Black (1967) shows that Dodgson (Carroll) treated the limited vote (a system in which each voter has k votes, where $k < m$ the number of seats to be filled, and the top k vote-getters are elected) as, in effect, a 2-person zero-sum game, and provided the maximin strategies players (political parties) should use in such a game. Thus Carroll was implicitly using game theory over 40 years before game theory was invented! In this context, it is not surprising that only one contemporary of Carroll was able to make any sense of Carroll's work (see Black, 1967: 17 n.1) and that it has languished in complete obscurity since 1885. Indeed, as Black confesses (1967:17, n.1) in earlier work (1958:181), he, too, had misjudged *The Principles of Parliamentary Representation.* As Black (1967:9-16) notes, Dodgson (1884) is also important in introducing a measure of the "number of voters unrepresented" and a method of calculating the expected proportion of unrepresented votes by using a prior probability distribution of the distribution of party strength. Inspired by Black's (1967) analysis, Mitchell (1976) discusses the fit between the Carroll game theory model and partisan campaign decisions in late 19th-century Great Britain, concluding that the model fits available evidence on party strategies rather well.

[63]For Black's own views that the connections between the theory of committees and elections and the work of Arrow are somewhat different, see Black (1969b) and especially Black (1972). In the latter unpublished article Black rejects as inappropriate the requirement that a committee decision rule satisfy Arrow's criterion of independence of irrelevant alternatives (reliance on pairwise choice). Unfortunately, Black's disagreement with Arrow is not fully clear to me, since there is some confusion in the literature over what "independence of irrelevant alternatives" is supposed to mean, and the discussion in Arrow (1963) is misleading in important ways. (See Plott (1976) for a helpful analysis of where the problem lies.)

Third, Black (1949b) offers a proposal of his own for a scheme of party list P.R. which would use rank-order data rather than simply first-place preference data to determine the seat proportions of the parties. In this article (1949b) on proportional representation Black also provides some insightful observations on the arbitrariness of the single transferable vote transfer procedures (Black, 1949b:336), on the merits of the greatest remainder vs. D'Hondt method of list P.R. (Black, 1949b:337),[64] and on the desirability of allowing voters (rather than parties) to specify the order in which party candidates will be selected (Black, 1949b:337-338). However, Black's own (1949b) proposal for a list P.R. system with panachage requires using statistical techniques to find the best-fitting parameter approximations to an overdetermined set of linear equations, and has little to recommend it, either in terms of comprehensibility for the ordinary voter or of practicality. It is omitted from Black (1958).

Subsequent to Black (1949b), there has been a great deal of important work done on the theoretical foundations of proportional representation, including some research (Brams, 1975) which extends and clarifies results on the limited vote in Black (1967), and some which introduces new indices similar in spirit to Carroll's index of nonrepresentation (Loosemore and Hanby, 1971; Rae, Hanby and Loosemore, 1971; Grofman, 1975).[65]

10. Rediscovering the Condorcet Jury Theorem

Consider a group of individuals confronting a choice between two alternatives. Let v_i be the competence of the ith member, i.e., the probability that he will "correctly" choose the superior alternative. When group members are assumed to be equally competent (i.e., $v_i = v$ for all i), Condorcet (1785) demonstrated that, if we weight type I and type II errors equally, then the quorum rule which maximizes the probability of a correct group choice is simple majority, and the accuracy of the group's judgment approaches one as the group's size increases, provided that $v > \frac{1}{2}$. This intriguing result, a variant of the "law of large numbers," which has come to be known as the Condorcet jury theorem (Grofman, 1975), is familiar to some 19th century scholars (most notably Poisson, 1837; see also references in Black, 1958:160-163), but has been "lost" for most of this century. Black (1958:164-165) restates Condorcet's results[66] and then shows (1958:165-173) how Condorcet sought (without great success) to extend the theorem to the multi-alternative case.

The problem that Condorcet was dealing with can be phrased roughly as follows: "For voters of equal judgmental competence, what is the voting rule that maximizes the likelihood that the committee decision (from among a set of $m \geqq 3$ alternatives) will be the 'correct' one?" Unfortunately, Condorcet (1785) provides an example to show that, in the multialternative cases, for some values of v, a Condorcet winner may be less likely to be the "correct" choice than another dominated alternative (see Black, 1958:169-170); and shows more generally that, in the absence of a Condorcet winner, which candidate from the top cycle has the highest probability of being the "correct" choice cannot be established as long as v is unknown.[67] With this failure of exact probabilistic methods,[68] Condorcet turns to what he refers to as "straightforward" reasoning (simple raisonnement) which leads him to assert that the right candidate to elect is the majority winner. In the three-alternative case with majority preferences aPbPc, Black paraphrases Condorcet as follows:

> There seems to be no argument at all in favor of c and the choice is between a and b. The argument in favor of b would have to run: we have reason to believe both that b is better than c and that b is better than a. The second of these propositions, however, is untrue, or at any rate has a probability of less than ½ in its favour; and this leaves the case for b very weak. Since we are making a choice between those two candidates and the proposition 'a is better than b' is more probable than the proposition 'b is better than a', we ought to elect a. (Black, 1958:170)

While no modern work that we are aware of has been done in extending the Condorcet jury theorem to the multialternative case, a great deal of recent attention has been paid to it (and to the somewhat more complex model proposed

[64] Black (1949b) does not use the term "D'Hondt method" and does not seem to be very familiar with the various types of list P.R. (e.g., D'Hondt, St. Lague, Modified St. Lague), but what he recommends is, we believe, mathematically identical to that method.

[65] We shall not attempt to summarize this recent literature on proportional representation here since elsewhere (Grofman, 1975) we review it in great detail and discuss other work on the impact of electoral systems, including both theoretical and empirical research on the alternative vote, cumulative voting, weighted voting, seats-votes relationships and the impact of the electoral college. Important work which has taken place since that review essay was written and not otherwise mentioned previously includes Balinksi and Young (1977a, 1977b, 1978, 1979, 1980, forthcoming), Sankoff (1971), Quandt (1971), Liphart (1977), Niemi and Deegan (1978), Still (1980, forthcoming), and Grofman and Scarrow (1980, forthcoming).

[66] While Black's mathematics is quite clear, he may be faulted for never clearly explaining in words what the theorem says, or indeed, ever referring to it as a theorem. In addition to Black (1958), useful discussions of the jury theorem and related work of Condorcet include Guilbaud (1952), Grainger (1956), Barry (1965), Gillispie (1972), and Baker (1976). See also Smoke and Zajonc (1962), Kuflik (1977).

The Condorcet theorem has recently been rediscovered (Kazmann, 1973) without awareness of its historical roots (see Grofman, 1975).

[67] However, Condorcet shows that it may be possible to eliminate some alternatives in the top cycle as clearly less likely than others to be the "correct" choice (Black, 1958:171-172).

[68] We believe that Condorcet (and Black) are being unnecessarily pessimistic about the limited value of the probabilistic approach, since under the specified assumptions, examples like the one given by Condorcet (see Black, 1958:169, Figure 16) are, we believe, a highly implausible occurrence. What Condorcet does is to assess the probabilities, given some observed distribution of votes, that each of the given alternatives is the "correct" choice on the assumptions that choice is made with respect to perceived superiority and that all voters are alike in their judgemental capacities. These assumptions give rise to a probabilistic version of single-peaked preferences. Condorcet's three-alternative example is implausible, since the distribution of votes has b overwhelmingly perceived to be superior to c, yet perceived to be close in worth to a, which is perceived as close in worth to b. Even though judgments are probabilistic, such striking disparities in collective judgments are, we suspect, extremely unlikely under the specified assumptions as to the similarities of individual choice. This is a matter which we hope to pursue in subsequent research.

by Poisson, 1837) in terms of applications to actual jury decision-making where jurors confront a simple choice between voting acquittal or voting conviction.[69]

One common formulation is the two-parameter model analyzed at length by Gelfand and Solomon (1973, 1974, 1975) and by Grofman (1974a, 1980). In this model

P_G = probability that the accused is guilty

p = probability that a juror will not vote for an incorrect verdict.

Gelfand and Solomon (1973, 1977) use this model to assess, from a societal standpoint, the implications of varying jury size for the expected percentage of "correct" verdicts and for the expected percentage of convictions—under the assumption of a process whereby the "effective" jury decision rule is simple majority.

Intuition would suggest that the larger the jury size, the less likely is conviction. Intuition, however, can be misleading. Under a wide range of parameter assignments under the specified assumptions, Gelfand and Solomon (1974:36) find the difference in the expected conviction rate of six-member and twelve-member juries to be negligible.

Gelfand and Solomon (1973, 1974) have fitted this two-parameter model to data on (unanimous verdict) criminal trials in Brooklyn and Chicago in the 1950's, drawn from Kalven and Zeisel (1966), and also to data on (7/12ths or, in some years, 8/12ths verdict) criminal trials in France in the 1830's drawn from Poisson (1837). They find values of $P_G = 0.64$ and $p = 0.75$ for the French data and $P_G = 0.70$ and $p = 0.90$ for the U.S. data. Thus, the success of the American criminal justice system in weeding out innocents prior to trial does not appear to be much better than that of the French criminal justice system of over a century ago; however, American jurors appear to be more "discriminating" than their French counterparts of last century. Of course, as Gelfand and Solomon (1974:36) point out, "more analysis and interpretation would be required before one could place strong faith in these conclusions."

Grofman (1974a, 1979) has looked at jury decision-making under the assumption of a K/N effective decision rule, where K is the number of votes which is (de facto) necessary for conviction and where N is jury size. Using data

[68]Exploration of the idea of group judgmental competence outside the jury context includes Grofman (1975), which provides a simple formula to calculate exactly how many individuals of competence v-x are equivalent to one individual of superior competence v; Margolis (1976), which looks at the effects of adding less competent members to a group; Grofman (1978), which looks at the case where not all v are equal, and in particular at the probability that the best member of the group is more likely to be correct than the group majority; and Miller (1980), which looks at partisanship and probabilistic electoral choice. There is also a considerable amount of work in progress on related topics. For example, Arnold Urken (personal communication, 1979) is investigating the unequal v_j case and comparing committee and subcommittee judgments; Lloyd Shapley has proved an elegant and quite general result on optimal weighted voting rules for group decision-making (Shapley and Grofman, 1980; and Scott Feld, personal communication, 1979) is looking at how the Condorcet jury theorem may be used to score exams for which the answer key has been lost, and how optimal cheating rules may be devised.

Other work which is in a similar spirit, although not traceable directly to the Condorcet jury theorem, includes Rae (1969), Taylor (1969), Schofield (1971, 1972), Niemi and Weisberg (1972), Badger (1972), Curtis (1972). See also Luce (1959) and Intriligator (1973).

on twelve-member (unanimous verdict) criminal trials in New York City 1971-1972, Grofman (1976a) finds an 8/12ths model to offer the best, but still rather unsatisfactory, fit. Fitting the unanimity model to this New York City data leads to parameter estimates of $P_G = 0.64$, $p = 0.996$. Thus, the unanimity model is seen to require an *absurdly* high mean juror discrimination capability, and this provides us with reason for rejecting it in favor of some form of group conformity process model.

In an extension of the two-parameter model, Grofman (~~1974~~, 1980) has examined the consequences of varying jury size and "effective" majority requirements in terms of a criterion parameter which is used to differentially weigh the desirability of "convicting the guilty" and "freeing the innocent." Grofman (1974) shows that unanimity may be desirable as the effective decision rule even for cases where "convicting the guilty" is regarded as more desirable than "freeing the innocent," provided mean juror discrimination capability is low and/or the pretrial screening process is extremely ineffective in "weeding out" the innocent. Grofman (1980) has also shown that, for jurors who would be willing to see as many as r guilty defendants set free rather than allow one innocent person to be convicted, that the decision rule which minimizes expected juror disappointment in the verdict outcome is an $\frac{r}{r+1}$ rule.

In another extension of the two-parameter model, Gelfand and Solomon (1974, 1975) and Grofman (1974) have each independently proposed a three-parameter model where P_G is as before but where

P_{GG} = probability that a juror will find a guilty defendant guilty

P_{II} = probability that a juror will find an innocent defendant innocent.

Gelfand and Solomon (1974) fit this model to data drawn from Kalven and Zeisel (1966) previously analyzed via the two-parameter model and find $P_G = 0.66$, $P_{GG} = 0.90$, and $P_{II} = 0.92$. Comparing these values to their previous findings of $P_G = 0.70$, $p = 0.90$, they conclude that the three-parameter model offers little improvement over the two-parameter model, given the fact that P_{GG} and P_{II} do not appear to differ much for the juror population under investigation. Alternative techniques for parameter estimations for this model, developed in Gelfand and Solomon (1975), lead them to reaffirm this conclusion. Finally, Grofman (1980) has used the two-parameter model to show, for reasonable assumptions as to the nature of the jury persuasion process, that majority rule juries are superior to unanimous juries in terms of minimizing *both* type I and type II errors. This is, of course, a quite counterintuitive result.

For a detailed and insightful review of the literature on jury decision-making models see Penrod and Hastie (1979).[70]

[70]For alternative though closely related approaches, see Nagel and Neef (1975), Klevorick and Rothschild (1979).

III. Conclusions: Duncan Black, Mathematical Political Science, and the History of Ideas.

Black's work in rediscovering and making intelligible the characteristically cryptic writings of earlier theorists has been a labor of love and of erudition. He has rescued from obscurity works which were misunderstood in their own time and would have been largely unknown in ours had he not saved them from what we now can recognize to be undeserved neglect.[71] Black's historical investigations allow us to see contemporary work on the theory of committees and elections as the continuation of an intellectual tradition of long and extremely distinguished lineage.

The debt we owe Black for this historical scholarship is immense, but it is outweighted in importance by the magnitude of Black's own research contributions to the pure theory of politics in calling attention to the importance of "procedures, agendas, and the search for principles which govern the behavior of voting processes" (Plott, personal communication, February 5, 1980). It can with considerable justice be claimed that Black was the first "public choice" economist.[72] Of him (as of Condorcet and Carroll) it can be said, "And here be giants."

[71]There had been a rediscovery of Condorcet's work in France prior to Black (1958) (see esp. Guildbaud, 1952; Grainger, 1956; cf. Laboule, 1939); while Borda's work was rediscovered by de Grazia (1953). Dahl (1956) in his discussion of the paradox cites Nanson (1882), which he learned about from Arrow (1951), and also Borda (1781), which he learned about from de Grazia (1953). Nanson's work is also reviewed in Baldwin (1929), a publication with which few American scholars are likely to be familiar. The historical summary in Black (1958) is, however, unique not only in covering the otherwise completely unknown work of Dodgson, Galton, and Laplace, but also in clearly and simply reformulating the ideas of Borda, Condorcet, Dodgson, etc., in a comprehensible fashion and as part of a lively and coherent essay in intellectual history. The historical contributions of Black (1958) are further extended in Black (1966, 1967, 1970). Of all the early scholars whom Black might have discussed, Poisson (1837) is probably the only one whose work could be regarded as important enough to have deserved review.

Plott (personal communication, February 5, 1980) has called attention to Black's work as an important precursor to the literature on path independence.

[72]I am not alone in viewing Black as the founder of public choice. Kenneth Arrow (1969:105) has pointed out that Black's work in the 40s synthesizes a number of important traditions (including work on voting systems, public finance, and applications of marginal utility theory), and Arrow asserts that it "began the continuous and now flourishing tradition" which "seeks to explain the political process in terms of the rational behavior of its participants" (Arrow, 1969:105, with some change of word and sentence ordering).

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