Metadata of the chapter that will be visualized online

Chapter Title	The Shapley–Owen Value and the Strength of Small Winsets: Predicting Central Tendencies and Degree of Dispersion in the Outcomes of Majority Rule Decision-Making	
Copyright Year	2014	
Copyright Holder	Springer International Publishing Switzerland	
Author	Family Name	**Feld**
	Particle	
	Given Name	**Scott L.**
	Suffix	
	Division	Department of Sociology
	Organization	Purdue University
	Address	West Lafayette, IN, 47907, USA
	Email	sfeld@purdue.edu
Author	Family Name	**Godfrey**
	Particle	
	Given Name	**Joseph**
	Suffix	
	Organization	WinSet Group, LLC
	Address	6107 Arlington Blvd #E, Falls Church, VA, 22044, USA
	Email	jgodfrey@winset.com
Corresponding Author	Family Name	**Grofman**
	Particle	
	Given Name	**Bernard**
	Suffix	
	Division	Department of Political Science
	Organization	Institute for Mathematical Behavioral Sciences, Center for the Study of Democracy, University of California
	Address	Irvine, CA, 92697-5100, USA
	Email	bgtravel@uci.edu
Abstract	Drawing on insights about the geometric structure of majority rule spatial voting games with Euclidean preferences derived from the Shapley–Owen value (Shapley and Owen, Int J Game Theory 18:339–356, 1989), we seek to explain why the outcomes of experimental committee majority rule spatial voting games are overwhelmingly located within the *uncovered set* (Bianco et al., J Polit 68:837–50, 2006; Polit Anal 16:115-37, 2008). We suggest that it is not membership in the uncovered set, *per se*, that leads to some alternatives being much more likely to	

become final outcomes of majority decision-making than others, but the fact that alternatives differ in the size of their winsets. We show how winset size for any alternative is a function of its squared distance from the point with minimal win set, and how this point, referred to by Shapley and Owen (Int J Game Theory 18:339–356, 1989) as the *strong point*, is determined as a weighted average of voter ideal points weighted by their Shapley–Owen values. We show that, in experimental voting games, alternatives with small winsets are more likely to be proposed, more likely to beat a status quo, and are more likely to be accepted as the final outcome than alternatives with larger winsets.

The Shapley–Owen Value and the Strength of Small Winsets: Predicting Central Tendencies and Degree of Dispersion in the Outcomes of Majority Rule Decision-Making

Scott L. Feld, Joseph Godfrey, and Bernard Grofman

1 Introduction

There are many different models of pivotal voting power that have been proposed. Most of these fall into the category of what are called *a priori* power scores. These are ones where some distribution of feasible outcomes is assumed and the probability of a given voter being pivotal is calculated wrt to that sample space based on that voter's (relative) weight in some particular voting game. One of the least known, but potentially most important of the power measures that are not *a priori* is the *Shapley–Owen value* (Shapley and Owen 1989), which is based on a uniform distribution of alternatives over a two dimensional (or higher) issue space, with the voters taken to be points embedded in that issue space, and with voter preferences customarily, for simplicity, taken to be Euclidean. The Shapley–Owen value is not regarded as an *a priori* power score since the power score (SOV) assigned to voters

The third-named author's contribution to this research was supported by the Jack W. Peltason (Bren Foundation) Endowed Chair at UCI and under Social Sciences and Humanities Research Council of Canada (SSHRCC) Grant #410-2007-2153 to study "Political Competition" (co-PIs: Stanley Winer, and J. Stephen Ferris).

S.L. Feld
Department of Sociology, Purdue University, West Lafayette, IN 47907, USA
e-mail: sfeld@purdue.edu

J. Godfrey
WinSet Group, LLC, 6107 Arlington Blvd #E, Falls Church, VA 22044, USA
e-mail: jgodfrey@winset.com

B. Grofman (✉)
AQ1 Department of Political Science, Institute for Mathematical Behavioral Sciences, Center for the Study of Democracy, University of California, Irvine, CA 92697-5100, USA
e-mail: bgtravel@uci.edu

R. Fara et al. (eds.), *Voting Power and Procedures*, Studies in Choice and Welfare,
DOI 10.1007/978-3-319-05158-1_14,

is a function of exactly where in the issue space these voters are located, not simply on voter weights in the voting game. In this essay we will draw on insights from the SOV in simple majority rule spatial voting games in two dimensions where voters have Euclidean preferences.

There is a long history of inquiry into the stability and predictability of majority rule processes in contexts where alternatives can be taken as points in a multidimensional issue or policy space (e.g., Plott 1967; Kramer 1972; Shepsle and Weingast 1981; Miller et al. 1989; Koehler 2001). It is widely understood that there are generally no equilibria, and that there are usually majority preferred paths that can lead from any position to any other position in the issue space (Mckelvey 1979, 1976). As Bianco et al. (2006, 2008) note, the findings of this earlier literature have been widely interpreted to imply that one can neither expect stability nor predictability of outcomes in spatial voting situations. Nevertheless, there are also incontrovertible empirical findings from experimental committee voting games that committee voting processes do reach stopping points that are not merely random. And, when we look at real world data in situations where we can estimate the ideological location of both voters and observed outcomes, e.g., wrt to voting processes such as those in the U.S. Congress or the U.S. Supreme Court, we again find a far from random pattern of outcomes relative to the distribution of estimated legislator voter ideal points,

Like Bianco et al. (2004, 2006, 2008), Schofield (1993, 1995a, b, 1999) and earlier work such as Ferejohn et al. (1984), we suggest that, even there is no core to the voting game, while all outcomes may be possible, some are more likely than others. In particular, as we shall see, the Shapley–Owen value, and insights derived from it about the underlying geometric structure of majority rule preferences, can aid us in identifying where outcomes of majority rule spatial voting games are most likely to be found.

Bianco et al. (2004, 2006, 2008) focus on the set of points in the *uncovered set* (Miller 1980, 1983) as the likely outcomes of majority rule voting processes over a "king of the hill" type agenda.[1] The *uncovered set* is the set of points such that no alternative in the set has another alternative that is both majority preferred to it and majority preferred to all point that it defeats. Another way of defining the uncovered set is as the set of points that beat all other points either directly or at one remove.[2] Thanks to new developments in computer software (Bianco et al. 2004;

AQ2

[1]In a "king of the hill" agenda, there is a prevailing alternative and in pairwise fashion, some new alternative (proposal) is matched against the present "king of the hill." If the new alternative fails to receive a majority against the present king of the hill, then the process continues with a second, third, etc. alternative being proposed. If the new alternative defeats the present king of the hill, then it becomes the new king of the hill, and the process continues Either the agenda for this process is finite, e.g., a given status quo which enters the last vote, or there is some procedure for invoking cloture, so that voters can stop the process once they find an "acceptable" king of the hill.

[2]The work of Schofield we have previously cited uses a solution concept called the *heart*, which seems very appropriate for weighted voting coalition games with a limited number of players, such as multiparty cabinet formation games, where ideal points are to a large extent a matter of common

Godfrey 2007) it is now possible to identify the location of solution concepts such as the uncovered set even for games with large numbers of actors, even though an analytic solution for the uncovered set is known only for the three-voter case (Feld et al. 1987; Hartley and Kilgour 1987).

Looking at results over a 20 year period of experimental research, Bianco et al. (2006) show that around 90 % of all the observed outcomes in nearly a dozen five person experimental committee voting games lie within the uncovered set. The Bianco et al. (2006) article represents, in our view, a major theoretical breakthrough in that, until their work, except in games where there was a core (where the prediction that outcomes in experimental (committee voting) games would tend toward the core was strongly supported), there simply was not a satisfactory game theoretic model to predict where outcomes would lie in committee voting games. The absence of satisfactory theory for non-core situations is highlighted in the discussion of results in Fiorina and Plott (1978) for their non-core game, and similar language is found in later experimental work on committee voting games up until very recently. Moreover, in our view, committee voting experiments trailed off after the late 1980s in part because of the absence of reliable theory that could be tested and further extended, while experimental work focused on areas, such as the study of auctions, where theory with real predictive bite was much better established.

Building on the Bianco et al. (2006) work on the predictive power of the uncovered set, we take a different, albeit related, tack. We will look for mechanisms that can explain why outcomes of committee voting games are likely to be in the uncovered set. This exploration will take us away from the uncovered set, *per se*, to look, instead, for even more general features of the structure of majority rule in the spatial voting context, features that we will demonstrate to be directly linked to the Shapley–Owen value.

The "winset" of a point is the set of other points that a majority of voters prefer to that point. Saying that there are no equilibria is equivalent to saying that all points have non-empty winsets. Nevertheless, the sizes of those winsets can vary widely. The simple intuition we propose is that, at least for king of the hill type agendas (and probably far more broadly) the size of a point's winset is a major determinant of whether a point is likely to be proposed, whether it is likely to be majority adopted, and whether is likely to be a stopping point of the voting process.

First, when a point is proposed, points with smaller winsets are more likely to be adopted because, by definition, points with smaller winsets are majority preferred to more possible status quos than other points. Second, points with smaller winsets are more likely to become the stopping point of the voting game because a majority is likely to recognize that it is difficult and unlikely for them to find and adopt a position that would be better for them, because points with small winsets will, by definition, offer few such alternatives that can defeat them and so, in a king of the

knowledge. Schofield (1999) shows that the uncovered set is a subset of the heart. In this essay we focus on committee voting games rather than coalition games, and we will draw our comparisons to the uncovered set rather than the heart.

hill type agenda, proposals to replace them are likely to fail (or at least to require a time consuming search).

This line of argument gives rise to two very straightforward hypotheses about majority rule processes.

Empirical Hypothesis 1 Outcomes of a majority rule process are more likely to be points with smaller winsets than points with larger winsets.

Empirical Hypothesis 2 Outcomes of a majority rule process tend to center around the point with the smallest winset.

Understanding the practical implications of these hypotheses for majority rule voting games requires us to draw on theoretical insights from Shapley and Owen AQ3 (1989) about the Shapley–Owen value. In particular.

Theoretical Proposition 1 (Shapley and Owen 1989) For Euclidean majority rule voting games in two dimensions, the point with the smallest winset, referred to by Shapley and Owen (1989) as the *strong point*, is located at the weighted average of the voter ideal points in the game, where the weights are simply each voter's Shapley–Owen value, i.e., the proportion of median lines on which each voter is pivotal.

The *strong point* is the spatial analogue of the *Copeland winner* in finite AQ4 alternative games, i.e., the point that is defeated by the fewest other points (Straffin and Philip 1980).

Theoretical Proposition 2 (Shapley and Owen 1989) For Euclidean majority rule voting games in two dimensions, for alternatives located along any ray from the strong point, the size of winsets increases with distance from the strong point. Even more specifically, the winset of any point has an area equal to the area of the winset of the strong point plus pi times its squared distance from the strong point.

Corollary to Theoretical Proposition 2 The larger the winset of the strong point itself, the less the *relative* difference in winset size as the distance to the strong point increases.

From this theoretical result about differences in win-set size as we move away from the strong point tied to the size of the strong point's winset, we are led to our third empirical hypothesis—one that allows for a prediction about comparisons of results across different experimental voting as a function of the location of the voter ideal points in those games and the concomitant size and win set area of the strong point.

Empirical Hypothesis 3 The smaller is the winset of the strong point itself, the closer, *ceteris paribus*, will be the outcomes of a majority rule process to the strong point, and the lower the variance of the observed outcomes.

In particular, in the limit, when the strong point shrinks to a single point, the AQ5 core, with an empty winset, we expect outcomes to be very close to this core—

a result which conforms to what has previously been found in studies involving experimental committee voting games with a core.

In the next section: (1) We provide some illustrative examples of winsets for majority rule processes in two-dimensional spatial contexts. (2) We formally describe the majority rule processes that have been used in experiments, and show the geometry of some of the spatial voting games used in these experiments. (3) We show that analyses of outcomes of these experiments are consistent with our theoretical predications. (4) We analyze not just final outcomes but also intermediate proposals in a few of these experiments to illustrate the plausibility of our proposed links between winset size and final outcomes. In the concluding discussion, after summarizing our empirical findings, we consider how our theory helps to explain the prediction success of the "uncovered set" as a solution concept.

2 Theoretical Properties of Winsets and Empirical Results About the Predictive Power of Winset Size for Outcomes in Experimental Games

2.1 Winsets in Majority Rule Processes in Two Dimensions

We begin consideration of winsets in spatial voting situations with a simple example. Suppose that there is a group of faculty deciding on the requirements for their graduate program in the context of a two-dimensional space, where the horizontal dimension is the number of requirements, and the vertical dimension is the relative emphasis on qualitative versus quantitative research approaches. For the purposes of illustration, Fig. 1a shows the current set of graduate program requirements as the origin in the graph. Suppose that there are three voters: "quant" who prefers more extensive quantitative requirements with an ideal point to the upper right; "qual" who prefers somewhat more extensive qualitative requirements with an ideal point to the lower right; and "easy" who just prefers less extensive requirements than currently in place, with an ideal point somewhat to the left. The status quo point in this example has a winset as shown in the Fig. 1a.

As noted earlier, Shapley and Owen (1989) show that the point with the winset of minimum area, called the *strong point*, is a weighted average of the locations of all the voter ideal points, where the weights are determined by the ranges over which each ideal point determines the boundaries of the winset, i.e., the range of angles over which the voter is pivotal. When there are only three voter ideal points, the angles of the triangle connecting those points turn out to be the relevant weights. Thus, the voter who subtends the largest angle has the single greatest influence on the location of the strong point. For the example presented above, the relevant angles are highlighted in Fig. 2 below.

It can be seen that qual has the largest angle; easy has the next; and quant has the smallest in this situation. The point with the smallest winset is the locations

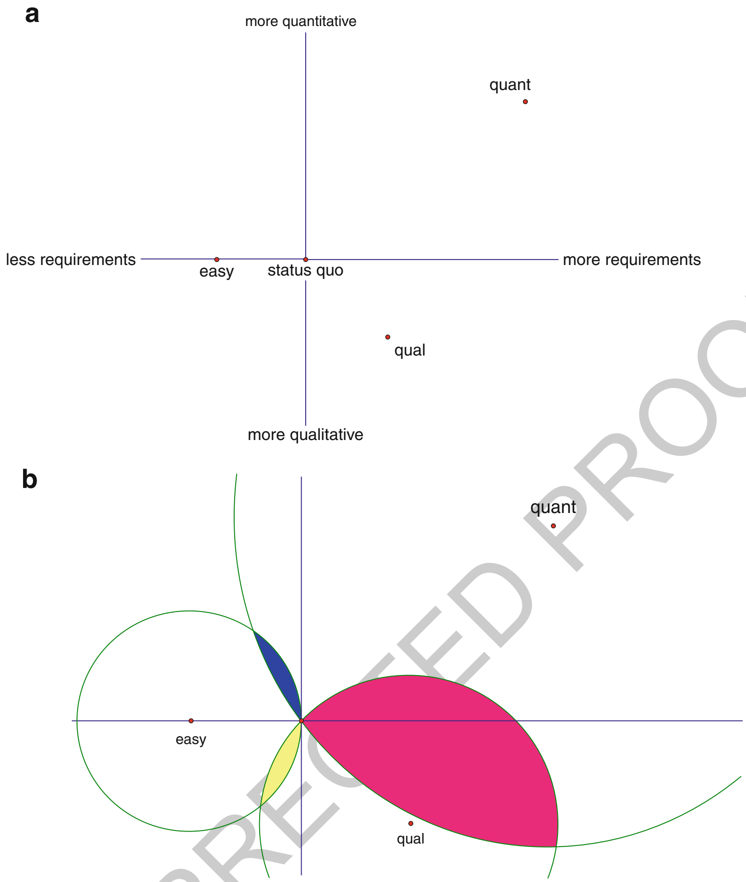


Fig. 1 A three voter game and the Winset of a status quo point in the game. (**a**) A three voter example of a committee voting game. (**b**) The Winset of the status quo in (**a**)

of the voter ideal points weighted by their angles. In this case the weights turn out to be approximately 0.5, 0.3, and 0.2, respectively. That weight-averaged point "strong point" is shown with its winset in Fig. 3a. As noted earlier, Shapley and Owen (1989) further prove that the winsets of points increase in size directly as the squared distance from the strong point. Thus, points that are equidistant from the strong point have equal size winsets. The circles in Fig. 3b indicate sets of points with equal size winsets.

Recall that our theoretical prediction is that points with smaller winsets are more likely to be the endpoints of sequential majority rule voting processes. Therefore, our theory implies that points that are closer to the strong point are more likely to be outcomes of majority rule processes than points further away, *ceteris paribus*. Furthermore, since winset sizes increase equally in all directions from the strong

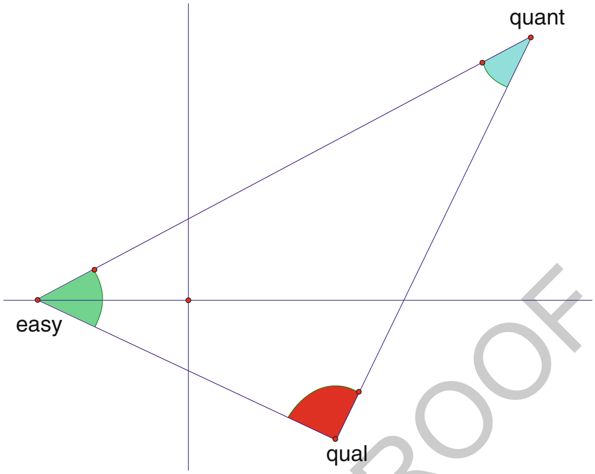


Fig. 2 Angles used as weights to determine the location of the strong point in the three-voter example of Fig. 1

point, outcomes are equally likely to be in any direction from the strong point, and are therefore expected to center around the strong point.[3]

In these games the process of voting is carried out by independent individuals with incomplete information. In particular, unlike the situation with voting games over a finite set of alternatives, sophisticated voting in the sense of Farquharson (1969), which requires the ability to use backward induction to identify sophisticated strategies (Mckelvey et al. 1978), is simply not possible. Furthermore, in the usual setting of experimental games, where information about voter ideal points is withheld from the players, individuals cannot calculate the sizes of winsets and, in any case, have no particular personal interest specifically in supporting points with smaller winsets. Their interests are in supporting points that are closer to their ideal points. Any strategy they employ that goes beyond that is likely to be highly variable from individual to individual. Nevertheless, we suggest that the effect of the "social" process driven by the preferences of the voters for outcomes closer to themselves is likely to result in overall outcomes being points with smaller winsets. The key intuition is that points with small(er) win-sets, once chosen, are "hard(er)" to defeat.

AQ6

At the same time, we recognize that there are many points with very similarly sized winsets. For example, points near the strong point have winsets that are only very slightly larger than the winset of the strong point. It is unlikely that any process that is driven by the relative sizes of winsets can make fine distinctions. Thus, points very near to the strong point are essentially equally likely to be the end of the process

[3]Note that points outside of the Pareto Set are unlikely to be outcomes of these types of processes even when they have relatively small winsets, and the strong point can sometimes be near the boundary of the Pareto Set. Consequently, we expect that situations where the strong point is close to the boundary of the Pareto Set will be exceptions to our general expectation that the strong point will be central among the outcomes.

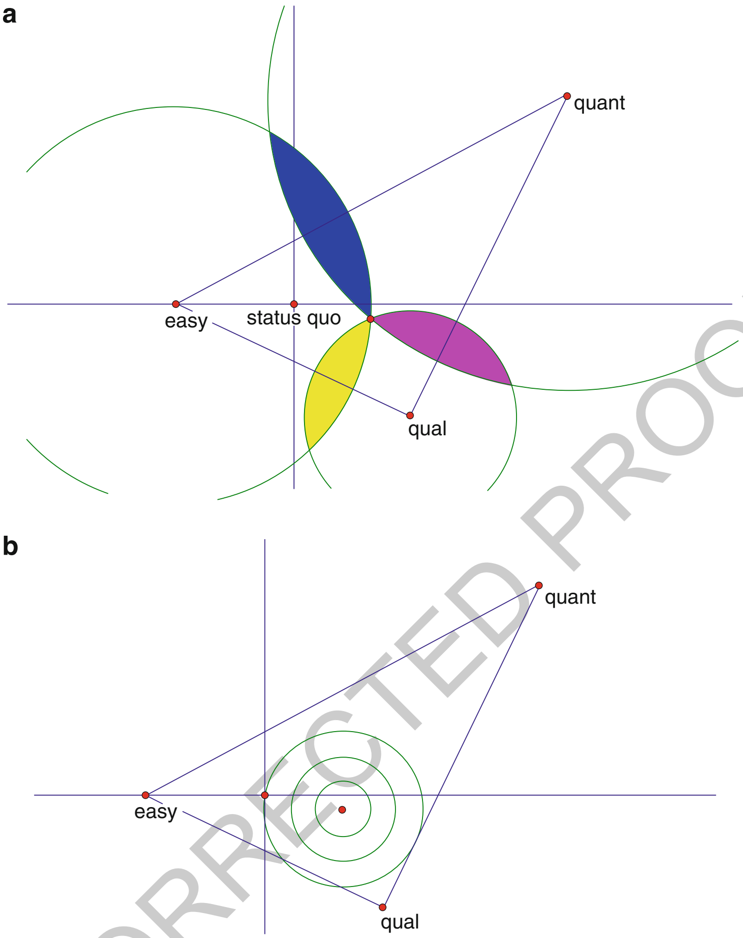


Fig. 3 The alternative with smallest Winset, and Winset sizes as we move away from that point in the three voter example of Fig. 1. (**a**) The Winset of the strong point. (**b**) Circles with equal size Winsets around the strong point

as the strong point itself. Also, as winset size increases with distance from the strong point, there will also be increasing numbers of points with winsets of each larger size**;** specifically, each concentric circle around the strong point can be thought an iso-winset line, and the further out circles are larger and contain more points. Thus, as we move further from the strong point, there are more points to choose from even though each is chosen with a lower probability.

It is impossible to specify a general functions describing the expected distance of outcomes from the strong point, based upon the countervailing effects of the increasing availability of points and the declining likelihood of each particular point, because the specific likelihood of outcomes at each distance will depend upon many factors determining the declining marginal utility to the particular actors, their ability to detect such differences under the particular circumstances of play, and whatever might influence their willingness and abilities to do anything about it, among other things.

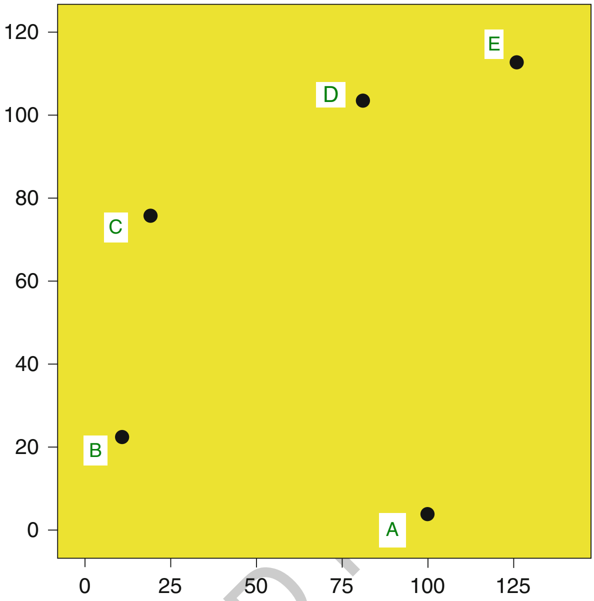
Nonetheless, we can hyopothesize with some confidence that, *ceteris paribus*, the faster that winsets in any given voting game grow with distance from the strong point, the closer that the outcomes will be, on average**,** to the strong point. We suggest that the relevant rate of growth is relative, i.e. the proportionate increase, rather than the absolute increase. The absolute size of the winsets always increase as pi d-squared, where d is the distance from the strong point. However, the relative importance of those increases declines with the size of the smallest winset, the winset of the strong point itself. This leads us to our third theoretical prediction, namely that the variation among the outcomes around the strong point will increase with the size of the winset of the strong point. When the winset of the strong point is very small, then the outcomes are likely to cluster relatively closely to the strong point. However, when the winset of the strong point is large, then the variation of the outcomes will be larger.

The variance around the strong point will depend, however, not only upon the size of the winset of the strong point, but also upon other factors previously suggested**,** e.g., the many factors determining the declining marginal utility to the particular actors, their ability to detect such differences under the particular circumstances of play, and whatever might influence their willingness and abilities to do anything about it, among other things. For example, we might expect that anything that makes players more nervous or impatient will lead them to be more willing to accept and vote to end at outcomes that they would not otherwise accept—that would imply greater variation in outcomes overall. But there is a strong *ceteris paribus* operating in our analyses: we simply do not know enough about how the selection of players, experimental instructions, and the play of the game itself may affect variation of outcomes around the strong point**.**

2.2 Majority Rule Committee Voting Experiments in Spatial Contexts

Bianco et al. (2006, 2008) review most of the experiments conducted by a variety of different researchers on majority rule processes in a spatial context that involve committee decision making. Fiorina and Plott published their classic experiment in 1978, and established the paradigm for subsequent experiments. The typical procedures in these experiments have involved five subjects voting for a point on a two-dimensional map. A session begins with a status quo point determined by the researcher. By various procedures, a proposal for an alternative positions in the space. Is proposed. Then, the group votes on whether or not to replace the current status quo with the proposed alternative. If a majority of the voters prefer

Fig. 4 The "Skew Star" five voter experimental game



the alternative, then that alternative becomes the new status quo point. Then, a new alternative is proposed and voted upon, etc. The process ends when a voter proposes stopping, and a majority of the voters approve of stopping at that point. Researchers have modified these procedures and limited the alternatives that can be proposed in various ways for various theoretical purposes, but the basic procedures have been similar in several experiments.

Figure 4 shows a typical situation, this one drawn from experiments conducted by Laing and Olmstead (1978). They called this their "Skew Star" situation.

In the game shown in Fig. 4, a hypothetical sequence of votes might move the status quo around the space as shown in the following hypothetical example (see Fig. 5).

Just as with the simple three voter situation in the previous section of this paper, each possible status quo point in this five voter game has a winset that consists of a set of petals, where each petal is the set of points that are preferred to the status quo point by some majority of the voters. Some points have smaller winsets than others as shown in Fig. 6. For example, point S has a smaller winset than T.

There is a single point with the smallest winset, and winset size increases with the squared distance to that strong point. The strong point and its winset is shown in Fig. 7.

AQ7 The actual set of outcomes for the 18 experimental runs conducted by Laing and Olmstead (1978) for this game are shown in Fig. 8, which also shows, for comparison purposes, the mean location of the outcomes in the game as well as the location of the strong point.

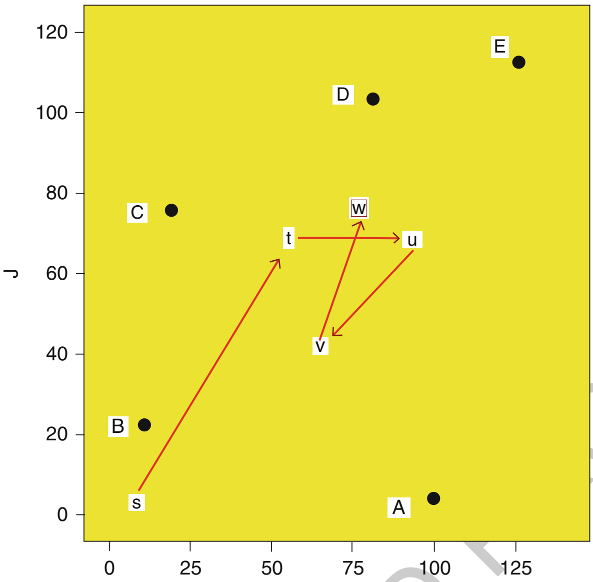


Fig. 5 A hypothetical trajectory of votes in the Skew Star game of Fig. 4

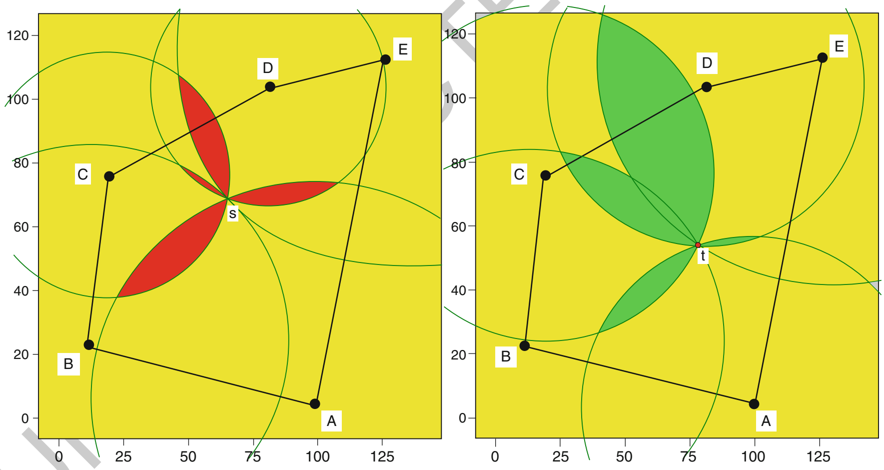


Fig. 6 Winsets of two points in the Skew Star game of Fig. 4

As noted previously, we have proposed three empirical hypotheses about the outcomes of spatial voting games, which we may summarize as below:

1. Points with smaller winsets are more likely outcomes than points with larger winsets.
2. The outcomes will tend to center on the strong point.

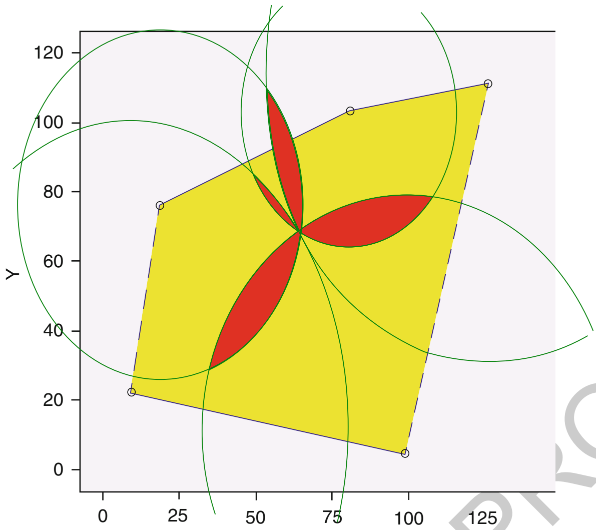


Fig. 7 The Winset of the strong point in the Skew Star game shown in Fig. 4

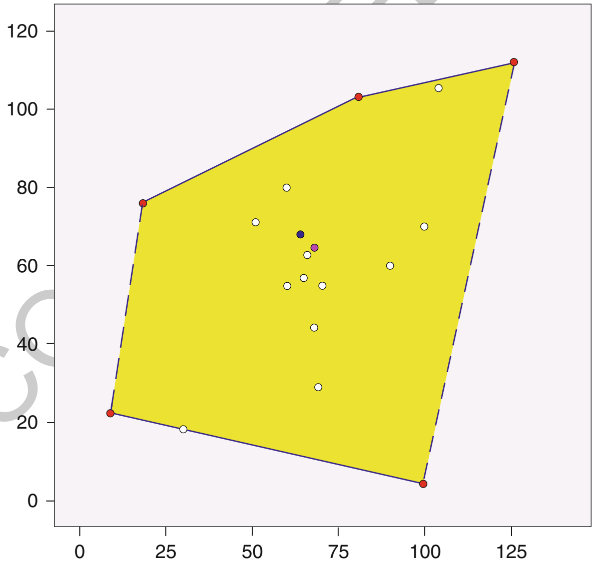


Fig. 8 The experimental outcomes in the Skew Star game shown in Fig. 4, showing the mean location of the outcomes in *black* and the location of the strong point in *red* (color figure online)

3. The variance of outcomes from the strong point will be smaller the smaller is the size of the winset of the strong point.

AQ8 Inspection of Figs. 9 and 10 illustrates support for the first two of our hypotheses. First, as expected, the outcomes are disproportionately clustered near to the strong point, i.e., are among the points with smallest winsets.

Second, also as expected, the strong point is located relatively centrally among the experimental outcomes because winset sizes increase symmetrically around the strong point with distance from the strong point. Nevertheless, the outcomes include some that are fairly far away from the strong point.

Our theory (Hypothesis 3) also suggests that there will be greater variation in ourtcomes when the size of the winset of the strong point itself is relatively large. In the Skew Star game of Fig. 4, as we will see when we present comparisons of this game to other games later in the paper, the win set of the strong point in the game is relatively large with respect to the Pareto set (see Fig. 8), and so winset sizes will rise only slowly with distance from the strong point, and thus, as expected, we get a fairly considerable scatter of outcomes around the strong point (see Fig. 9).

These detailed data from this one experiment are presented merely to illustrate how we use experimental findings to examine and test the implications of our hypotheses. The data for all the relevant experiments are analyzed more systematically in the next section.

2.3 Testing Our Hypotheses Using a Large Body of Data on Experimental Outcomes in Committee Voting Games

Using data from the same experiments reanalyzed by Bianco et al. (2006) and additional experiments that Bianco et al. (2008) conducted themselves, we reanalyze games used in 18 different experiments by seven different teams of researchers. Two of these games were initially used by Mckelvey and Ordeshook under several
AQ9 different experimental conditions and then used again by Endersby (1993) under other experimental conditions. For our present purposes, we combined all the data collected for the same games even if conducted under different conditions by different experimenters. Thus, we were able to reanalyze the results from a total of ten different games.

The experiments whose outcomes are used here were conducted for a variety of different specific purposes, including testing different solution concepts under somewhat different structural conditions. For the present purposes we are ignoring the relatively small differences in experimental procedures among the experiments to focus on the overall tendencies that emerge even when there are some potentially confounding differences among the experimental protocols.

First, we find that the mean positions among the experimental outcomes in all of these games are very close to the strong points of the games. Table 1 shows the coordinates of the mean outcome compared with the coordinates of the strong point

Table 1 Mean outcomes as compared to the strong point in the game

Game	Mean outcome	Strong point	Pareto set area
Bianco 1	(38,73)	(41,78)	2450
Bianco 2	(67,20)	(67,19)	1673
Fiorina_Plott_1978	(45,63)	(47,62)	4647
Laing_Olmsted_1978_A2_The_Bear	(83,57)	(80,55)	7875
Laing_Olmsted_1978_B_Two_Insiders	(66,34)	(61,35)	5106
Laing_Olmsted_1978_C1_The_House	(76,55)	(90,53)	9311
Laing_Olmsted_1978_C2_Skew_Star	(69,67)	(64,69)	8190
McKelvey_Ordeshook_Winer_1978	(84,122)	(88,115)	10845
PH	(57,36)	(58,36)	2753
PHR	(67,34)	(70,32)	2763

along with the area of the Pareto set so that we can see how close the strong point is to the mean outcome in each game relative to the size of the Pareto set.[4]

We see from Table 1 that mean outcomes are close to the strong point in nine of ten games, and close in one dimension, but not so close in the second dimension, in the remaining game. For each of the ten games, we did significance tests to determine whether the mean for the x coordinate was statistically significantly different from the x coordinate of the strong point, and similarly for the y-coordinate. For these 20 significance tests, two of them were statistically significant, which is close to what would be expected by chance alone ($p = 0.05$) if the means for the populations were exactly at the strong points.

Second, not only do the outcomes tend to tend to be close to the strong point, on average, but they are also close to the strong point when we think of distance in terms of the size of the Pareto set. In general, the distance between the means of the outcomes and the strong points are less than 2 % of the sizes of the Pareto sets.

Third, outcome variance tends to be related to the size of the winset of the strong point, as we theoretically predicted. When the winset of the strong point is smaller, relative to the size of the Pareto set, then there is less variation in the outcomes around the strong point (again relative to the size of the Pareto set), as is shown in Table 2.

Over this small set of ten games the correlation between outcome variance in the game and the size of the winset of the strong point in the game is +0.21. It would be much stronger except for two outliers. The Bianco two game has a strong point with

[4]Consider, for example, the strong point (shown in black) in Fig. 10. It is very close to the mean location of the outcomes (shown in pink) when we think of closeness relative to the spread of the voter ideal points. The Pareto set in these situations is the convex figure that is enclosed by all the lines between the voter ideal points. For any point outside the Pareto set, the voters always unanimously prefer some other point inside the Pareto set. Consequently, voters generally have little reason to ever propose alternatives outside of the Pareto set, and they rarely do so. Consequently, the effectiveness of prediction should be considered with respect to proposals in the Pareto set (shaded yellow in Fig. 10).

AQ11

Table 2 Winset size and outcome variance

Game	Winset size	Variance in outcomes
Fiorina_Plott_1978	0.02	0.02
Bianco_2_2003	0.02	0.14
McKelvey_Ordeshook_Endersby_PHR	0.06	0.03
McKelvey_Ordeshook_Winer_1978	0.06	0.05
Laing_Olmsted_1978_C1_The House	0.06	0.15
McKelvey_Ordeshook_Endersby_PH	0.07	0.04
Laing_Olmsted_1978_C1_The Bear	0.11	0.06
Laing_Olmsted_1978_C1_Skew Star	0.11	0.11
Bianco_1_2003	0.12	0.1
Laing_Olmsted_1978_C1_Two_Insiders	0.15	0.11

a small winset, but has considerable variance in outcomes, and the Laing-Olmsted House game also has considerably more variation than would have been expected. If these two outliers are omitted, the correlation for the remaining eight games is 0.90.

Further examination of these two games with unexpectedly high variation in outcomes indicates that the high variation in each case arises from just a couple of extreme outlying outcomes. A closer examination of the data from the Bianco two game (where we were able to examine the whole process for each experimental run) indicates that the two extreme outcomes in that game occurred when the voters made the rare decision to stop immediately after accepting their first proposal. Such findings suggest that there may be idiosyncratic noise in any play of any particular play of a game with a small number of players. However, the rest of the pattern suggests that apart from such "noise", the outcomes are consistent with our theoretical expectations that there is generally less variance in outcomes when the strong point has a smaller winset.

2.4 Proposing, Adopting, and Stopping in Experimental Committee Voting Games

Unfortunately, the data on proposed and adopted points are not included in the published reports on any of the experimental studies used in the previous analyses. However, William Bianco and his colleagues (personal communication, 2009) have generously provided us with these data from their recent experiments with two particular experimental games. These data provide a relatively small number of cases, but are sufficient to provide some preliminary findings. We combine the results of the two games in Table 3 below. The starting point (also provided to us by Professor Bianco) is not included in the analyses shown in that table.

In both of these games, it is clear that the winset sizes of proposed points are considerably smaller than for other Pareto points, that winsets of adopted points

Table 3 Mean and standard deviations of Winset sizes for different sets of points

	Bianco game 1			Bianco game 2		
	Mean	SD	*n*	Mean	SD	*n*
Pareto points	1314	1463		2210	2118	
Proposed points	531	882	261	785	1499	185
Adopted points	386	396	147	446	770	91
Stopping points	246	236	28	244	641	28

are considerably smaller than other proposed points, and that stopping points have considerably smaller winsets than other adopted points. Even with these relatively small sample sizes, all of these differences are statistically significant (with $p \leq 0.05$ using 1-tail tests). Thus, as hypothesized, points with smaller winsets are more likely to be majority approved, and chosen as stopping points than other points in the Pareto. Moreover point with smaller winsets are also more likely to be proposed.[5]

3 Conclusions

3.1 Key Findings

Until Bianco et al.'s recent publications, previous research seems to have led researchers to the conclusion that there was no good theory to predict the outcomes of experimental committee voting games in two or more dimensions.[6] Bianco et al. reopened the question with their findings that nearly all outcomes in large body of experimental voting games fell within the uncovered set, and that such results were considerably more likely than would be expected by chance. Our approach to this same data has emphasized the predictive power of small win sets.

[5] It is important to recognize that, unless there is a core to the voting game, it need not be true that points that beat other points have smaller winsets than the points they beat. At the start of the process, when the points are relatively far out from the strong point, there is a tendency for the process to move inward. However, once the status quo is further in toward the strong point, there is no necessary expectation that further points will have smaller winsets. In fact, nearly all the points in the winsets of points close to the strong point have winsets larger than the strong point itself—consequently, if the process does not stop at the strong point, it necessarily moves to points with a larger winset than the strong point itself. Nonetheless, if the outcome is a point near the strong point this will be a point with a relatively small winset.

[6] On the other hand, there are models of spatially embedded coalition formation games and of party competition games that do generate empirically testable models that garnered considerable empirical support. Trying to reconcile the theoretical and empirical findings on committee voting, coalition formation, and party competition, however, takes us into issues well beyond the scope of this paper.

While predicting that points with smaller winsets are more likely outcomes does not provide any specific boundaries for the set of predicted outcomes, it does allow us to make some specific predictions. First, we predicted that points with smaller winsets are more likely outcomes than points with larger winsets. Second, since winset size is distributed symmetrically around the point with smallest winset, the strong point, we predicted that outcomes will center on the strong point. Third, since winset size increases as a specific monotonic function of distance from the strong point, and consequently relative winset size increases more slowly when the winset of the strong point itself is large (relative to the Pareto set), we predict that the outcomes will diverge further from the strong point in games when the winset of the strong point is large than in games when the winset of the strong point is small. Evidence from 17 experiments using 10 different experimental games confirms each of these predictions, and suggests that previous findings concerning the success of the uncovered set may result from the fact that points in the uncovered set tend to have small winsets.

Movement toward points with smaller winsets can be considered as a "centrifugal" force pulling outcomes toward the strong point. However, of course, we recognize that there are centripetal forces that may pull outcomes somewhat away from the strong point. For example, actors may tend to make proposals for alternatives that are close to their ideal points, and voters may accept outcomes that are good enough, even if not ideal. Also, any (minimal) winning coalition can exert total control of outcomes, and such coalitions may pull outcomes toward the hull of that coalition, which might not include the strong point. Furthermore, there may be confusions or misperceptions that also affect outcomes, and some voters may be more attentive to the voting process than others. Each of these aspects of the game (e.g., satisficing, coalition formation processes, variation in information levels or actor involvement) can pull outcomes away from the strong point. Moreover the specific voting rules (e.g., whether a defeated alternative can be reconsidered) and other features of the experiment (e.g., how much knowledge each voter has about the preferences of the other voters)[7] may matter a great deal, suggesting the desirability of additional experiments for a fixed set of voter locations to see how rules of the game and other context features matter for the mean and variance of outcomes and for speed of convergence. Nevertheless, we believe that size of win sets provides such a strong gravitational pull on outcomes that it will serve as a key theoretical tool for understanding and predicting outcomes and outcome trajectories not just in king of the hill spatial committee voting games, but also in a wider set of committee voting games, and in real world politics that can be modeled as voting over multidimensional issues

[7] To the extent that voters can develop a sense of the preferences of other players, points perceived as "more fair" may be more likely to be proposed and accepted as the final outcome, or perhaps, points that are perceived to be likely to defeat other alternatives, e.g., points on the boundary of a minimum winning coalition, may be more likely to be proposed (cf. the notion of the *competitive solution* in Mckelvey et al. 1978).

Table 4 Mean squared distance to the strong point from the Pareto, the uncovered set and the outcomes in the game that lie in the uncovered set

	Mean D squared		
Game	Pareto	UC	Outcome in UC
Bianco 1	1314	312	225
Bianco 2	2210	50	21
Fiorina_Plott_1978	2360	128	64
Laing_Olmsted_1978_A2_The_Bear	1569	853	323
Laing_Olmsted_1978_B_Two_Insiders	1304	654	454
Laing_Olmsted_1978_C1_The_House	2059	939	500
Laing_Olmsted_1978_C2_Skew_Star	1659	890	397
McKelvey_Ordeshook_Winer_1978	2364	1041	594
PH	601	281	93
PHR	597	290	100

3.2 *Reconsidering the Success of the Uncovered Set as a Predictor of Experimental Game Outcomes*

Part of the motivation for the present paper comes from recent publications by Bianco and colleagues reporting their findings that the *uncovered set* is a very successful solution concept for experimental committee voting games. While our empirical findings are only that we do as well in predicting outcomes with win-set size as we do with the uncovered set (taking our winset prediction set to be the same size as the uncovered set), we would argue that there are good reasons to prefer the winset explanation for observed experimental outcomes.

1. It is highly plausible that alternatives that defeat most other alternatives are less likely to be defeatable by a randomly chosen other alternative than points with a larger win set, and thus are more likely to end up ultimately chosen. In contrast there really is no comparably good "story" to explain the predictive success of the uncovered set.
2. Outcomes within the uncovered set have smaller winset sizes than general points in the Pareto set.[8] Table 4 shows that this is true for all of the experimental games that Bianco et al. analyzed and that we reanalyzed above. This results suggest that the correlation between being in the uncovered set and having a small winset may account for the predictive success of the uncovered set[9]

[8]The uncovered set consists of points with small winsets because points with large winsets are likely to be covered by some other point with a smaller winset (see Miller 2007).

[9]Bianco et al. (2004, 2006, 2008) note that when the uncovered set is large, the uncovered set can include most of the points in the Pareto Set, so predictions based on the uncovered set are not that specific, though they predict far better than chance. They are equally well aware that, in the unusual situations where the uncovered set is small, e.g. when there is a core, then some observed outcomes in experimental voting will not lie exactly at the core and thus will fall outside the uncovered set.

On the other hand, we would note that the evidence we have presented for the strong point determined by Shapley–Owen values being the center of the distribution of the observed outcomes of experimental spatial majority rule voting games must be interpreted with some caution, since there are other solution concepts that are also located very centrally in the Pareto set and very close to the strong point, e.g., the centroid of the uncovered set or the center of the *yolk*, the center of the smallest circles that touches all median lines.[10] The evidence presented in this paper does not really allow us to distinguish the hypothesis that points are centered around the strong point from the hypothesis that points are centered around the center of the yolk, or the centroid of the uncovered set.[11] It is only with further experimental work, especially work that allows us to examine what points are proposed as well as what points remain "king of the hill," that we will be able to devise critical tests among competing explanatory models.

AQ13

References

Bianco, W. T., Jeliazkov, I., & Sened, I. (2004). The uncovered set and the limits of legislative action. *Political Analysis, 12*, 256–276.

Bianco, W. T., Lynch, M. S., Miller, G. J., & Sened, I. (2006). A theory waiting to be used: a reanalysis of canonical experiments on majority rule decision-making. *Journal of Politics, 68*, 837–850.

Bianco, W. T., Lynch, M. S., Miller, G. J., & Sened, I. (2008). The constrained instability of majority rule. *Political Analysis, 16*, 115–137.

Endersby, J. W. (1993). Rules of method and rules of conduct: an experimental study on two types of procedure and committee behavior. *Journal of Politics, 55*, 218–236.

Farquharson, R. (1969). *Theory of voting*. New Haven: Yale University Press.

[10] Bounds on the uncovered set are often stated in terms of the center of the yolk (in two dimensions, the yolk is the smallest circle touching all median lines), e.g., in classic work, Mckelvey (1986) shows that the uncovered set must lie with 4 yolk radii of the center of the yolk, and this bound has been tightened by others (Feld et al. 1987; Miller 2007). Thus, it is natural to ask about the relationship between the center of the yolk and winset size, on the one hand, and the relationship between the center of the yolk and the strong point, on the other. Craig Tovey (personal communication, 2009), investigating a conjecture by Scott Feld, has recently proved a result closely related to the Shapley and Owen result bounding the size of winsets by the size of and distance to the strong point, namely that bounds on the size of the winset of any point can be stated in terms of the size of the winset of the strong point and the distance of the point to the center of the yolk. As a corollary, he also shows that the strong point can be no more than 2.16 yolk radii from the center of the yolk. Although that is the tightest bound known, we have not found any situations where the strong point is more than one yolk radius from the center of the yolk, and we believe it must be within the yolk in the three voter case. In the games reported on here, the yolk is considerably closer to the strong point than 2.16 yolk radii.

[11] Similarly, if voters focus their attention on one dimension at a time, resulting in an outcome at the generalized median, i.e., at the location of the respective median voter along each of two dimensions (Shepsle and Weingast 1981; cf. Feld and Grofman 1988), the generalized median cannot be very far away from the strong point.

Feld, S. L., & Grofman, B. (1987). Necessary and sufficient conditions for a majority winner in N-dimensional spatial voting games—an intuitive geometric approach. *American Journal of Political Science, 31*, 709–728.

Feld, S. L., & Grofman, B. (1988). Majority rule outcomes and the structure of debate in one-issue-at-a-time decision making. *Public Choice, 59*, 239–252.

Feld, S. L., Grofman, B., Hartley, R., Kilgour, M. O., & Miller, N. (1987). The uncovered set in spatial voting games. *Theory and Decision, 23*, 129–156.

Ferejohn, J. A., Mckelvey, R. D., & Packel, E. W. (1984). Limiting distributions for continuous state Markov voting models. *Social Choice and Welfare, 1*(1), 45–69.

Fiorina, M. P., & Plott, C. R. (1978). Committee decisions under majority rule: an experimental study. *American Political Science Review, 72*, 575–598.

Hartley, R., & Kilgour, M. (1987). The geometry of the uncovered set in the three-voter spatial model. *Mathematical Social Sciences, 14*(2), 129–155.

Koehler, D. (2001). Instability and convergence under simple-majority rule: results from simulation of committee choice in two-dimensional space. *Theory and Decision, 50*(4), 305–332.

Kramer, G. (1972). Sophisticated voting over multidimensional spaces. *Journal of Mathematical Sociology, 2*, 165–180.

Laing, J. D., & Olmstead, S. (1978). An experimental and game-theoretic study of committees. In P. C. Ordeshook (Ed.), *Game theory and political science* (pp. 215–281). New York: New York University Press.

Mckelvey, R. D. (l976). Intransitivities in multidimensional voting models and some implications for agenda control. *Journal of Economic Theory, 12*, 472–482.

Mckelvey, R. D. (1979). General conditions for global intransitivities in formal voting models. *Econometrica, 47*, 1085–1112.

Mckelvey, R. D. (1986). Covering, dominance, and institution-free properties of social choice. *American Journal of Political Science, 30*(2), 283–314.

Mckelvey, R. D., & Niemi, R. G. (1986). A multistage game representation of sophisticated voting for binary procedures. *Journal of Economic Theory, 18*(1), 1–22.

Mckelvey, R. D., Winer, M. D., & Ordeshook, P. (1978). The competitive solution for N-person games without transferable utility, with an application to committee games. *American Political Science Review, 72*(June), 599–615.

Miller, N. R. (1980). A new 'solution set' for tournaments and majority voting. *American Journal of Political Science, 24*, 68–96.

Miller, N. R. (1983). The covering relation in tournaments: two corrections. *American Journal of Political Science, 27*(2), 382–385.

Miller, N. R. (2007). In search of the uncovered set. *Political Analysis, 15*(1), 21–45.

Miller, N. R., Grofman, B., & Feld, S. L. (1989). The geometry of majority rule. *Journal of Theoretical Politics, 1*(4), 379–406.

Plott, C. R. (1967). A notion of equilibrium and its possibility under majority rule. *American Economic Review, 57*, 787–806.

Schofield, N. (1993). Political competition and multiparty coalition governments. *European Journal of Political Research, 23*(1), 1–33.

Schofield, N. (1995a). Democratic stability. In J. Knight & I. Sened (Eds.), *Explaining social institutions*. Ann Arbor: University of Michigan Press.

Schofield, N. (1995b). Coalition politics – a formal model and empirical-analysis. *Journal of Theoretical Politics, 7*(3), 245–281.

Schofield, N. (1999). The heart and the uncovered set. *Journal of Economics-Zeitschrift Fur Nationalokonomie Supplement, 8*, 79–113.

Shapley, L., & Owen, G. (1989). Optimal location of candidates in ideological space. *International Journal of Game Theory, 18*, 339–356.

Shepsle, K. A., & Weingast, B. (1981). Structure-induced equilibrium and legislative choice. *Public Choice, 37*(3), 503–519.

Straffin, J., & Philip, D. (1980). *Topics in the theory of voting*. Boston: Birkhauser.

AUTHOR QUERIES

AQ1. Please check if the affiliation details are presented correctly.
AQ2. "Godfrey, 2007" is cited in text but not given in the reference list. Please provide details in the list or delete the citation from the text.
AQ3. Please check this line "about the Shapley-Owen value. In particular." and give suggestion.
AQ4. The citation "Straffin, 1980" (original) has been changed to "Straffin and Philip 1980". Please check if appropriate.
AQ5. Please check if the edit made to the sentence "we expect outcomes to be very close" is fine.
AQ6. The citation "Mckelvey and Niemi, 1978" (original) has been changed to "Mckelvey et al. 1978". Please check if appropriate.
AQ7. The citation "Laing and Olmsted (1985)" (original) has been changed to "Laing and Olmstead (1978)". Please check if appropriate.
AQ8. Figures 9 and 10 are cited in the text but both captions and art are missing in the manuscript. Please check.
AQ9. The citation "Endersby" (original) has been changed to "Endersby (1993)". Please check if appropriate.
AQ10. Please check and confirm if the usage of the decimal commas in the values in Table 1 are fine as given.
AQ11. Please check if the edits made in Table 2 are fine.
AQ12. Please check if the edits made to the column head in Table 4 are fine.
AQ13. References "Feld and Grofman (1987)" and "Mckelvey and Niemi (1986)" are not cited in text. Please check.