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The impartial culture maximizes the probability of majority cycles

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Abstract. Many papers have studied the probability of majority cycles, also called the Condorcet paradox, using the impartial culture or related distributional assumptions. While it is widely acknowledged that the impartial culture is unrealistic, conclusions drawn from the impartial culture are nevertheless still widely advertised and reproduced in textbooks. We demonstrate that the impartial culture is the *worst case scenario* among a very broad range of possible voter preference distributions. More specifically, *any deviation from the impartial culture over linear orders reduces the probability of majority cycles* in infinite samples unless the culture from which we sample is itself inherently intransitive. We prove this statement for the case of three candidates and we provide arguments for the conjecture that it extends to any number of candidates.

1 Introduction

Much analytical/theoretical work on majority rule has been devoted to the probability of cycles occurring in samples (groups of people) of different sizes. The most widely used probability distribution of preferences to generate the random profiles has been the impartial culture. The *impartial culture* is a

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uniform distribution over linear orders [2, 3, 5, 7, 8, 13, 14] or weak orders [4, 12, 15, 18].[1] Besides classifying these papers according to the primitive type of preference relation they use, one could also group them based on whether they use analytical or numerical methods, whether they study finite or infinite samples, and which method of sampling they study. The impartial culture is an example of a *culture of indifference*, by which we mean that every pairwise majority (in the culture) is a majority tie.

While the main glory days of research on the Condorcet paradox may be a thing of the past, some researchers still occasionally generate a spark of interest on the topic outside the specialized social choice literature. One of the more salient examples is Jones et al. in the *American Political Science Review* [12] who run Monte Carlo simulations in which they draw random samples from a uniform distribution over weak orders. They conclude that making preferences more "realistic" by allowing for weak order preferences (instead of linear order preferences) reduces the probability of cycles dramatically. For instance, for three candidates and 1,000,001 voters this probability reduces to 0.056 from 0.088. The latter is the probability of a cycle in a very large or infinite sample from the impartial culture over linear orders [11].[2] Jones et al. argue that this more realistic representation of preferences leads to cycle probabilities that are more consistent with the fact that cycles are rarely encountered empirically. We show here that changing the distribution in *any fashion* (whether we call it "realistic" or not) away from an impartial culture over linear orders will automatically have the effect of reducing the probability of majority cycles in infinite samples as long as the underlying culture itself does not inherently have an intransitive weak majority preference relation. In other words, in all cases but the degenerate case that preassumes the Condorcet paradox from the start, the impartial culture over linear orders is the worst case scenario.

The paper is organized as follows. Section 2 provides some useful background information and a primer on our argument. Section 3 contains three theorems. Theorem 1 states the fact that if the weak majority preference of the culture is transitive then, for three candidates, the probability of cycles in infinite samples is nonzero only for a culture of indifference. The next step is to show that the impartial culture over linear orders maximizes the probability of cycles in the class of all cultures of indifference over weak orders, for three candidates. Theorem 2 proves this statement. Combining Theorems 1 and 2 yields Theorem 3 which can be summarized as follows: in the case of

[1] A linear order is a complete ranking of alternatives without ties. A weak order is a ranking of alternatives including possible ties. A strict weak order is the asymmetric part of a weak order (i.e., ties are not counted as a "preference both ways" but as a "lack of strict preference either way"). Every linear order is a strict weak order, but not conversely.

[2] We use the abbreviation *infinite sample* to refer to the limiting case of samples of size n as $n \to \infty$.

three candidates the impartial culture over linear orders maximizes the probability of cycles in infinite samples drawn from any culture over weak orders which does not have a weak majority intransitivity built in from the start. In Sect. 4 we state the conjecture that the same result holds regardless of the number of candidates, and we provide several arguments for its support. We summarize our conclusions in Sect. 5 and prove the theorems in the Appendix.

2 Background

Fifteen years before Jones et al's paper on the Condorcet paradox in samples from a weak order distribution, Fishburn and Gehrlein [4] analytically derived the probability of cycles for three, four and five candidates in infinite samples drawn from a mixture of the impartial culture over linear orders and impartial cultures over weak orders with a fixed number of ties.[3] From their formula it follows that the impartial culture over linear orders maximizes the probability of cycles among all mixtures of the impartial culture over linear orders and impartial cultures over weak orders with a fixed number of ties (for 3, 4, or 5 candidates).[4] Jones et al. [12] reported Monte-Carlo simulation results that correspond to the already mentioned analytical formulae and which expand these results to finite samples and more than five candidates (see also [16] for more details). Jones et al. conclude that for large samples ($n > 501$), "highly indifferent electorates are most likely to find Condorcet winners" in comparison to the impartial culture over linear orders (Jones et al. [12], p.141). The parallel finding for infinite samples is already stated in Fishburn and Gehrlein [4]: "... it is seen that the likelihood of the [Condorcet] paradox decreases as individual indifference increases" (Fishburn and Gehrlein [4], abstract).

Gehrlein [6, p.38] shows that the impartial culture over linear orders for three candidates maximizes the probability of cycles in samples drawn from any dual culture (here, only linear orders are possible, each linear order has the same probability as the reverse linear order).[5] Thus, for three candidates and linear order individual preferences, any deviation from a symmetric distribution (with respect to any relabeling of the candidates) also reduces the probability of cycles.

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[3] [4] has unfortunately been largely ignored in the literature. In particular, it is not cited in [12]. For an earlier study based on computer simulations, see also [1].

[4] Although all the necessary mathematics is present in Fishburn and Gehrlein [4] the final formula for the case of three candidates appears explicitly only in Lemma 1 of Gehrlein and Valognes [10].

[5] For three candidates, the dual culture is a culture of indifference over linear orders. Therefore, Gehrlein [6] proves that the impartial culture maximizes the probability of cycles compared to any culture of indifference over *linear orders* for three candidates.

All these results already suggest that the traditional impartial culture over linear orders is a worst case scenario in the sense that it maximizes the probability of cycles, at least in infinite samples.[6] However, all sampling results we are aware of only compare the impartial culture to particular other cultures of indifference and therefore are only special case results. The first question we pose is therefore whether the impartial culture maximizes the probability of cycles among all cultures of indifference. Our ultimate question in this paper is whether the impartial culture maximizes the probability of cyclic samples among *all* cultures that are not already cyclic to start with. To the best of our knowledge, the answer to that question has not been provided in the literature. While we will generalize dramatically beyond the usual distributional assumptions, we will continue to assume that the underlying culture has itself a transitive majority preference relation. Otherwise, large samples will simply reproduce the cycle that has been assumed in the first place. More specifically, if one were to draw random samples from a distribution which already contains majority cycles then random samples would display the same cycles with probability one as sample size approaches infinity.

Our main theorem proves that, in the case of three candidates, the impartial culture is indeed the worst case scenario in the sense that it maximizes the probability of majority cycles in infinite samples drawn from any culture with transitive weak majority preferences. (A culture has a *weak majority preference* of candidate a over candidate b if there are at least as many people who prefer a to b as people who prefer b to a.)

Results of this type are known for the dual culture [6] (this requires maximization over two variables) and for the mixture of the impartial culture over linear orders and the impartial culture over weak orders with exactly one tie [4, 10] (this requires maximization over a single variable). However, it is not obvious that the same result holds for all cultures of indifference over weak orders. In general, there are 13 possible weak orders with 4 restrictions, and therefore to find a distribution with a maximal probability of cycles in infinite samples we need to maximize over 9 variables. One restriction is that the sum of the probabilities of all 13 weak orders equals 1, and three more restrictions are imposed by the requirement that we are dealing with a culture of indifference, i.e. that all three pairwise majority preferences are ties.

We concentrate on the case of three candidates not only because of its analytical tractability. Transitivity is a property over triples: if the probability

[6] The existing research on the impartial anonymous culture (here, only linear orders are possible, each profile is equally probable) also supports this thesis: for three candidates and infinite sample size, the probability of cycles for the impartial anonymous culture is $\frac{1}{16}$ [7], which is less than the probability of cycles for the impartial culture over linear orders (0.088). Samples drawn from the maximal culture (here, only linear orders are possible, each linear order frequency in the profile is drawn independently from the uniform distribution) do not have a fixed number of voters and therefore are quite different from the random samples drawn from some particular distribution. The probability of cycles under the maximal culture condition is no less than $\frac{11}{120}$ [9].

of cycles is zero for each triple among m candidates then the majority relation over all m candidates is transitive with probability one. Having proven the theorem for the case of three candidates, we present strong intuitive arguments for the conjecture that the impartial culture over linear orders must be the worst case regardless of the number of candidates.

3 The impartial culture is the worst case for three candidates

Throughout the paper, we refer to a fixed finite set $\mathscr{C}$ of choice alternatives, candidates, parties or consumption bundles. Let $\mathscr{B}$ be a collection of strict weak orders over $\mathscr{C}$. For any two candidates a, b, and any voter with preference relation $B \in \mathscr{B}$, the standard notation aBb means that a person in the state of preference B finds that a is *better than* b. Let us denote the probability of each strict weak order $B \in \mathscr{B}$ in the culture by p_B. We use the word *population* in the statistical sense and in a fashion interchangeable with the word *culture*. Any given population probability distribution over $\mathscr{B}$ is conceptualized as a set of nonnegative parameters $p = (p_B)_{B \in \mathscr{B}}$ with the restriction that $\sum_{B \in \mathscr{B}} p_B = 1$. For example, if $B = abc$, a complete linear ranking, then p_{abc} denotes the probability that a person drawn at random from the culture has the linear preference order abc, i.e., this person likes a best, b second best and c least. To make the notation more readable, we write p_{aBb} for $\sum_{B \in \mathscr{R}} p_B$ where $\mathscr{R} = \{B \in \mathscr{B} \text{ such that } aBb\}$. So, in the special case when $\mathscr{B}$ is the set of complete linear orders of 3 candidates then $p_{aBb} = p_{abc} + p_{acb} + p_{cab}$. In the general case allowing for indifference, we write aEb if and only if neither aBb nor bBa holds. In other words, aEb denotes the situation where a and b are *equivalent* (hence the notation). A person in this state of preference has no strict preference either way, i.e., s/he is indifferent between the two. If a person is indifferent between a and b, but prefers both of them to c, we write $aEbBc$. Given any culture, captured by the probability distribution p over strict weak orders, we say that a is preferred to b by a *weak majority* in the culture if $p_{aBb} \geq p_{bBa}$. We use boldface to denote random variables and regular font to denote numbers. Suppose that we draw a random sample of size n with replacement from the distribution p. Analogous to the notation for p, we write $\mathbf{N}_{aBb}$ to denote the number of people who prefer a to b in such a random sample. We also write $P(\mathbf{N}_{aBb} > \mathbf{N}_{bBa})$ for the probability that a majority in the sample prefers a to b. We denote the probability of cycles in infinite samples drawn from a distribution p by $P_c(p)$.[7] For three alternatives a, b, c this probability is specified by the following function:

[7] Since the probability of a majority tie between the two alternatives in an infinite sample is zero (unless everybody is indifferent between these two alternatives), looking for majority intransitivities is equivalent to looking for majority cycles.

$$P_c(p) = \lim_{n\to\infty}\begin{pmatrix} P((\mathbf{N}_{aBb} > \mathbf{N}_{bBa}) \cap (\mathbf{N}_{bBc} > \mathbf{N}_{cBb}) \cap (\mathbf{N}_{cBa} > \mathbf{N}_{aBc})) \\ + P((\mathbf{N}_{aBb} < \mathbf{N}_{bBa}) \cap (\mathbf{N}_{bBc} < \mathbf{N}_{cBb}) \cap (\mathbf{N}_{cBa} < \mathbf{N}_{aBc})) \end{pmatrix}.$$

The first theorem can be summarized as follows: Suppose individual preferences are strict weak orders over three alternatives. Assume further that the population weak majority preferences are transitive. For instance, the impartial culture (and any culture of indifference) satisfies that criterion because its own majority preference relation is a complete tie, and thus transitive. We prove that in the limit (as $n \to \infty$), the probability of cyclical majority preferences is different from zero only if all three alternatives are majority tied in the underlying population. A similar result, based on more specialized assumptions, is given in [19].

Theorem 1. *Let $\mathscr{C} = \{a, b, c\}$ and let $p : B \mapsto p_B$ be a probability distribution on strict weak orders over $\mathscr{C}$. Suppose that the weak majority preference relation is transitive, i.e. for any relabeling $\{x, y, z\} = \{a, b, c\}$*

$$\left.\begin{matrix} p_{xBy} - p_{yBx} \geq 0 \\ p_{yBz} - p_{zBy} \geq 0 \end{matrix}\right\} \Rightarrow p_{xBz} - p_{zBx} \geq 0.$$

Then $P_c(p)$ is different from zero only if all pairs of candidates are majority tied:

$$P_c(p) > 0 \Rightarrow \begin{cases} p_{aBb} = p_{bBa}, \\ p_{bBc} = p_{cBb}, \\ p_{aBc} = p_{cBa}. \end{cases}$$

In other words, as sample size goes to infinity, cycles are possible only if samples are drawn from a culture of indifference.

The proof of this theorem is provided in the Appendix. The second theorem establishes that the impartial culture over linear orders maximizes the probability of cycles within the class of all cultures of indifference.

Theorem 2. *Let $\mathscr{C} = \{a, b, c\}$ and let $p : B \mapsto p_B$ be a probability distribution on strict weak orders over $\mathscr{C}$, satisfying a culture of indifference, i.e., $p_{aBb} = p_{bBa}$, $p_{bBc} = p_{cBb}$, and $p_{aBc} = p_{cBa}$. Then $P_c(p)$ reaches its maximum at the following values:*

$$p_{abc} = p_{acb} = p_{cab} = p_{bac} = p_{bca} = p_{cba} = \frac{1 - p_{aEbEc}}{6}; \quad 0 \leq p_{aEbEc} < 1,$$

i.e., for the impartial culture over linear orders plus "total indifference" (p_{aEbEc}).

The proof of this theorem is given in the Appendix. It is straightforward to show that, for $n \to \infty$, the probability p_{aEbEc} of the complete indifference relation does not influence the probabilities of possible majority preference relations (as long as $p_{aEbEc} < 1$).

Combining Theorems 1 and 2 we now prove that, for three candidates, the impartial culture over linear orders maximizes the probability of cycles compared to *any* culture over strict weak orders with a transitive weak majority preference relation.

Theorem 3. *Let $\mathscr{C} = \{a, b, c\}$ and let $p : B \mapsto p_B$ be a probability distribution on strict weak orders over $\mathscr{C}$. Suppose that the weak majority preference relation is transitive, i.e. for any relabelling $\{x, y, z\} = \{a, b, c\}$*

$$\left.\begin{array}{l} p_{xBy} - p_{yBx} \geq 0 \\ p_{yBz} - p_{zBy} \geq 0 \end{array}\right\} \Rightarrow p_{xBz} - p_{zBx} \geq 0.$$

Then $P_c(p)$ reaches its maximum at the following values of p:

$$p_{abc} = p_{acb} = p_{cab} = p_{bac} = p_{bca} = p_{cba} = \frac{1 - p_{aEbEc}}{6}; \quad 0 \leq p_{aEbEc} < 1,$$

i.e. for the impartial culture over linear orders plus "total indifference" (p_{aEbEc}).

Theorem 3 follows directly from Theorems 1 and 2. It is important to notice that Theorem 3 is valid only for an infinite sample size. For any specified (fixed) finite sample size it is easy to find a culture in which the probability of cycles in samples of that specified size is arbitrarily close to $\frac{1}{2}$ and therefore much higher than the probability of cycles for the impartial culture. For example, consider the following culture: $p_{abc} = \frac{1}{2} + 2\epsilon$, $p_{cab} = \frac{1}{4} - \epsilon$, $p_{bca} = \frac{1}{4} - \epsilon$. For any given n, choosing $\epsilon > 0$ small enough, we can force the probability of cycles to be close to $\frac{1}{2}$. Notice, however, that if ϵ is specified first then we can always find n large enough to make the probability of cycles as close to zero as we wish (Theorem 1).

4 Is the impartial culture the worst case for any number of candidates?

A rigorous proof of the corresponding theorem for more than three candidates seems to us intractable. Our arguments for the conjecture that the impartial culture over linear orders is the worst case for any number of candidates are the following.

– If we have m many candidates then the number of possible majority relations in infinite samples is $2^{m(m-1)/2}$ (because majority ties have probability zero). Only $m!$ of these majority relations are linear orders, i.e. transitive. Thus, there are $2^{m(m-1)/2} - m!$ possible intransitive majority relations in infinite samples. An intransitive majority relation can occur in an infinite sample only if there is a triple that has a majority cycle in that sample. On the other hand, the argument given in Theorem 1 implies that if only one triple of alternatives is majority tied in the underlying culture then a majority cycle in an infinite sample can occur only in that triple. Thus, if we want to give every one of the $2^{m(m-1)/2} - m!$ potential intransitivities a positive probability, we need a culture of indifference for each triple. From a combinatorics point of view, we therefore expect a culture of indifference over all candidates.

- It is plausible that, in order for the total probability of cycles to be maximized, the probability of cycles in any given triple should also be maximized. Together with Theorem 2 this suggests that we need to have the impartial culture over linear orders on each triple. This rules out all strict weak orders with one or more ties for each triple and consequently all strict weak orders with one or more ties. Consequently we conjecture that the probability of cycles is maximized for a population where only linear orders have positive probability.
- Because relabeling (permuting) the alternatives has no impact on the overall probability of cycles, a basic symmetry argument suggests that the extremum should also be reached at a culture that is itself symmetric with respect to any relabeling of the candidates. Therefore, we expect that the probability of cycles is maximized by a culture which is symmetric with respect to any relabeling of the candidates. In particular, consistent with this argument, in the case of three candidates the impartial culture is the unique symmetric distribution among dual cultures [6] and it maximizes the probability of cycles.
- The last two arguments together suggest that the probability of cycles is maximized for a symmetric (with respect to any relabeling of candidates) distribution over linear orders, that is, for the impartial culture over linear orders.

5 Conclusion

Several papers [4, 12, 18] have suggested that majority cycles (i.e., the Condorcet paradox) become less probable when replacing the impartial culture over linear orders by a "more realistic" culture which allows for individual indifference. We prove a much more general result for the case of three candidates, and conjecture the same for any number of candidates: Among all cultures over weak order preferences, that do not already presume some inherent weak majority intransitivity to begin with, the impartial culture over linear orders maximizes the probability of majority cycles in infinite samples. Even within cultures of indifference over linear order preferences, we can reduce the probability of majority cycles arbitrarily much by deviating from the impartial culture [6]. Therefore, reducing the probability of the paradox does not crucially rely on moving from linear order preferences to weak order preferences.

Overall, *any deviation*, "*realistic*" *or not*, from the impartial culture over linear orders will reduce the probability of the Condorcet paradox. In particular, reducing (or even eliminating) the Condorcet paradox, by itself, need not provide any new insight into the actual political and social process of collective decision making.

Furthermore, as Theorem 1 indicates, if the underlying culture contains no majority ties and no inherent majority intransitivities then infinite samples will have majority cycles with probability zero. By extension, sufficiently large electorates will encounter the Condorcet paradox with probability arbitrarily close to zero.[8] As a consequence, a sampling framework will allow for the Condorcet paradox in large voter profiles only in two circumstances: Either the paradox is already assumed to exist at the level of the culture, or the profiles are generated from a culture where sufficiently many pairs of alternatives are majority tied.

Appendix

Proof of Theorem 1

To simplify the exposition, we write $x \succ y$ for *x is majority preferred to y*. Suppose that among three alternatives a, b, c one pair is not majority tied, e.g. $a \succ b$ (i.e. $p_{aBb} > p_{bBa}$) in the population. Then, because the weak majority preference is transitive, either $a \succ c$ or $c \succ b$ (or both) in the population. Suppose that $a \succ c$. The probability of the cycle $a \succ b, b \succ c, c \succ a$ in a sample is bounded from above by $P(\mathbf{N}_{cBa} > \mathbf{N}_{aBc})$. The probability $P(\mathbf{N}_{cBa} > \mathbf{N}_{aBc})$ goes to zero as sample size goes to infinity, because $p_{aBc} > p_{cBa}$ by assumption. (This type of analysis using bounds on the probability of a particular majority relation is described in detail elsewhere [17].) Similarly, the probability of the cycle $b \succ a, a \succ c, c \succ b$ in a sample is no more than $P(\mathbf{N}_{bBa} > \mathbf{N}_{aBb})$ which also goes to zero as sample size goes to infinity, because $p_{aBb} > p_{bBa}$ by assumption.

Therefore, if the probability of cycles is greater than zero as sample size approaches infinity then the distribution of individual preferences in the population that governs the sampling process has to satisfy a culture of indifference. ■

Proof of Theorem 2

Let us spell out the probability of cycles for an infinite sample size. Using the multivariate normal approximation of the multinomial distribution we get the following expression for $P_c(p)$ in infinite samples drawn from a culture of indifference:

[8] For example, if there are m many candidates, and if the pairwise margin is no less than 10 percent for any pair of candidates (i.e., for any pair a, b, the difference between the number of people who prefer a to b and those who prefer b to a is no less than 10 percent of all voters), then an electorate of 951 voters is sufficient to avoid a Condorcet cycle with probability at least $1 - (0.001)\frac{m(m-1)}{2}$. This and other examples are discussed in more detail elsewhere [17].

$$P_c(p) = \frac{1}{4} - \frac{1}{2\pi}\left(\arcsin\left(\frac{\omega_{ab,ac}}{\sigma_{ab}\sigma_{ac}}\right) + \arcsin\left(\frac{\omega_{ab,bc}}{\sigma_{ab}\sigma_{bc}}\right) + \arcsin\left(\frac{\omega_{ac,bc}}{\sigma_{ab}\sigma_{bc}}\right)\right),$$

where $\omega_{ab,ac}$ and σ_{ab} for any alternatives a, b, c are given by

$$\begin{aligned}\omega_{ab,ac} &= p_{aBb\cap aBc} - p_{aBb\cap cBa} - p_{bBa\cap aBc} + p_{bBa\cap cBa}\\ &= p_{aBbBc} + p_{aBbEc} + p_{aBcBb} - p_{cBaBb} - p_{bBaBc}\\ &\quad + p_{bBcBa} + p_{bEcBa} + p_{cBbBa};\end{aligned}$$

$$\sigma_{ab} = \sqrt{1 - p_{aEb}} = \sqrt{1 - p_{aEbBc} - p_{cBaEb} - p_{aEbEc}}.$$

This formula generalizes the representation 1) for the probability of cycles drawn from a mixture of impartial cultures over linear orders and over weak orders with exactly one tie [4] and 2) for the probability of cycles in samples drawn from a dual culture [6] to any culture of indifference over weak orders. To find p at which the probability of cycles reaches a maximum, we need to find a minimum of the function

$$F = \arcsin\left(\frac{\omega_{ab,ac}}{\sigma_{ab}\sigma_{ac}}\right) + \arcsin\left(\frac{\omega_{ab,bc}}{\sigma_{ab}\sigma_{bc}}\right) + \arcsin\left(\frac{\omega_{ac,bc}}{\sigma_{ac}\sigma_{bc}}\right).$$

Introducing new variables

$$\sigma_3 = \frac{\sigma_{ab}}{\sqrt{1 - p_{aEbEc}}}, \quad \sigma_2 = \frac{\sigma_{ac}}{\sqrt{1 - p_{aEbEc}}}, \quad \sigma_1 = \frac{\sigma_{bc}}{\sqrt{1 - p_{aEbEc}}},$$

$$\omega_1 = \frac{\omega_{ab,ac}}{1 - p_{aEbEc}}, \quad \omega_2 = \frac{\omega_{ab,bc}}{1 - p_{aEbEc}}, \quad \omega_3 = \frac{\omega_{ac,bc}}{1 - p_{aEbEc}},$$

means that we need to find

$$Min\left(F = \arcsin\left(\frac{\omega_1}{\sigma_2\sigma_3}\right) + \arcsin\left(\frac{\omega_2}{\sigma_1\sigma_3}\right) + \arcsin\left(\frac{\omega_3}{\sigma_1\sigma_2}\right)\right)$$

such that $\omega_1 + \omega_2 + \omega_3 = 1$, with $|\omega_i| \le 1$ and $0 < \sigma_i \le 1$.

Now we are going to show that F reaches its minimum at

$$\omega_i = \frac{1}{3}, \sigma_i = 1, \quad i = 1, 2, 3.$$

From the restrictions on ω_i it follows that if one of the ω_i is negative then the other two are positive. Suppose that $\omega_3 < 0$. Because $\arcsin(x)$ is monotonically increasing in x, we see that $\frac{\partial F}{\partial \sigma_3} < 0$. Then $\sigma_3 = 1$, because we are looking for a minimum. Substituting $\omega_3 = z\sigma_1\sigma_2$ we can express F as

$$\begin{aligned}F &= \arcsin\left(\frac{\omega_1}{\sigma_2\sigma_3}\right) + \arcsin\left(\frac{1 - \omega_1 - z\sigma_1\sigma_2}{\sigma_1\sigma_3}\right) + \arcsin(z)\\ &= \arcsin\left(\frac{\omega_1}{\sigma_2\sigma_3}\right) + \arcsin\left(\frac{1 - \omega_1}{\sigma_1\sigma_3} - \frac{z\sigma_2}{\sigma_3}\right) + \arcsin(z).\end{aligned}$$

From this representation it is clear that $\frac{\partial F}{\partial \sigma_1} < 0$, and $\sigma_1 = 1$. Writing

$$F = \arcsin\left(\frac{1 - \omega_2 - z\sigma_1\sigma_2}{\sigma_2\sigma_3}\right) + \arcsin\left(\frac{\omega_2}{\sigma_1\sigma_3}\right) + \arcsin(z)$$

we get $\frac{\partial F}{\partial \sigma_2} < 0$ and $\sigma_2 = 1$. So we have proven the following necessary condition for F to reach its minimum:

$$\sigma_1 = \sigma_2 = \sigma_3 = 1.$$

Now the problem is reduced to the following: Find

$$Min(F = \arcsin(\omega_1) + \arcsin(\omega_2) + \arcsin(\omega_3))$$

such that

$$\omega_1 + \omega_2 + \omega_3 = 1, \quad |\omega_i| \leq 1.$$

Substituting the constraint on $\omega_1 = 1 - \omega_2 - \omega_3$, we can write

$$F = \arcsin(1 - \omega_2 - \omega_3) + \arcsin(\omega_2) + \arcsin(\omega_3).$$

Thus,

$$\frac{\partial F}{\partial \omega_3} < 0 \Leftrightarrow 1 - \omega_2 - \omega_3 > \omega_3,$$
$$\frac{\partial F}{\partial \omega_3} > 0 \Leftrightarrow \omega_3 > 1 - \omega_2 - \omega_3.$$

Since the minimum has to be at the point where the derivative changes its sign from negative to positive, we conclude that

$$\omega_3 = \frac{1 - \omega_2}{2} > 0.$$

Thus, in order for F to reach its minimum, it has to be that $\omega_i > 0$, $\forall i$. Since the function $\arcsin(x)$ is increasing and convex in x for $x \geq 0$, F reaches its minimum for $\omega_1 = \omega_2 = \omega_3 = \frac{1}{3}$. Solving this condition for p_B (from the equations for ω_i and σ_i) we get the proof of the theorem: the probability of cycles reaches its maximum for

$$p_{abc} = p_{acb} = p_{cab} = p_{bac} = p_{bca} = p_{cba} = \frac{1 - p_{aEbEc}}{6}; \quad 0 \leq p_{aEbEc} < 1. \quad \blacksquare$$

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